

# CENTER OF THE UNIVERSAL ASKEY–WILSON ALGEBRA AT ROOTS OF UNITY

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**ABSTRACT.** Inspired by a profound observation on the Racah–Wigner coefficients of  $U_q(\mathfrak{sl}_2)$ , the Askey–Wilson algebras were introduced in the early 1990s. A universal analog  $\Delta_q$  of the Askey–Wilson algebras was recently studied. For  $q$  not a root of unity, it is known that  $Z(\Delta_q)$  is isomorphic to the polynomial ring of four variables. A presentation for  $Z(\Delta_q)$  at  $q$  a root of unity is displayed in this paper. As an application, a presentation for the center of the double affine Hecke algebra of type  $(C_1^\vee, C_1)$  at roots of unity is obtained.

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## 1. INTRODUCTION

Throughout this paper an algebra  $\mathcal{A}$  is meant to be an associative algebra with unit and let  $Z(\mathcal{A})$  denote the center of an algebra  $\mathcal{A}$ .

Fix a complex scalar  $q \neq 0$ . In [36] Zhedanov proposed the Askey–Wilson algebras which involve five extra parameters  $\varrho, \varrho^*, \eta, \eta^*, \omega$ . Given these scalars the *Askey–Wilson algebra* is an algebra over the complex number field  $\mathbb{C}$  generated by  $K_0, K_1, K_2$  subject to the relations

$$\begin{aligned} qK_1K_2 - q^{-1}K_2K_1 &= \omega K_1 + \varrho K_0 + \eta^*, \\ qK_2K_0 - q^{-1}K_0K_2 &= \omega K_0 + \varrho^* K_1 + \eta, \\ qK_0K_1 - q^{-1}K_1K_0 &= K_2. \end{aligned}$$

These algebras are named after R. Askey and J. Wilson since the algebras can also describe a hidden relation between the three-term recurrence relation and the  $q$ -difference equation of Askey–Wilson polynomials [1]. Under the mild assumptions  $q^4 \neq 1$ ,  $\varrho \neq 0$  and  $\varrho^* \neq 0$  substitute

$$K_0 = -\frac{\sqrt{\varrho^*}A}{q^2 - q^{-2}}, \quad K_1 = -\frac{\sqrt{\varrho}B}{q^2 - q^{-2}}, \quad K_2 = \frac{\omega}{q - q^{-1}} - \frac{\sqrt{\varrho\varrho^*}C}{q^2 - q^{-2}}$$

into the defining relations of the Askey–Wilson algebra. The resulting relations become that each of

$$(1) \quad A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \quad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \quad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}$$

is equal to a scalar. By interpreting the elements in (1) as central elements, it turns into the so-called *universal Askey–Wilson algebra*  $\Delta_q$  [33]. Let us denote  $\Delta = \Delta_q$  for brevity.

Let  $\alpha, \beta, \gamma$  denote the central elements of  $\Delta$  obtained from multiplying the elements (1) by  $q + q^{-1}$ , respectively. Motivated by Zhedanov [36, §1], the distinguished central element

$$(2) \quad qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma$$

is called the *Casimir element* of  $\Delta$ . For  $q$  not a root of unity, the center of  $\Delta$  has been shown in [33, Theorem 8.2] to be the four-variable polynomial ring over  $\mathbb{C}$  generated by  $\alpha, \beta, \gamma$  and the Casimir element (2). The inspiration of our study on  $Z(\Delta)$  at roots of unity comes from

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the quantum group  $U'_q(\mathfrak{so}_3)$ . The quantum group  $U'_q(\mathfrak{so}_n)$  [9] is not Drinfeld–Jimbo type but plays the important roles in the study of  $q$ -Laplace operators and  $q$ -harmonic polynomials [17, 27],  $q$ -ultraspherical polynomials [31], quantum homogeneous spaces [26], nuclear spectroscopy [10],  $(2+1)$ -dimensional quantum gravity [24, 25] and so on. For  $n = 3$  the quantum group is exactly the Askey–Wilson algebra with  $q^4 \neq 1$ ,  $q = 1$ ,  $q^* = 1$ ,  $\eta = 0$ ,  $\eta^* = 0$ ,  $\omega = 0$ . According to [27, §4] the Casimir element of  $U'_q(\mathfrak{so}_3)$  is defined to be

$$(3) \quad q(q^2 - q^{-2})K_0K_1K_2 - q^2K_0^2 - q^{-2}K_1^2 - q^2K_2^2.$$

As far as we know, Odesskii [29, Theorem 4] first found three additional central elements of  $U'_q(\mathfrak{so}_3)$  at roots of unity defined as follows. Assume that  $q$  is a primitive  $d^{\text{th}}$  root of unity and set

$$\bar{d} = \begin{cases} d & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \text{ is even.} \end{cases}$$

Denote by  $\mathbb{Z}$  the ring of integers and by  $\mathbb{N}$  the set of the nonnegative integers. For each  $n \in \mathbb{N}$  define

$$(4) \quad T_n(X) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \left( \binom{n-i}{i} + \binom{n-i-1}{i-1} \right) X^{n-2i}.$$

Here  $\binom{n}{i}$  for  $n \in \mathbb{N}$  and  $\binom{-1}{i}$  are interpreted as 0 and 1, respectively. Note that  $\frac{1}{2}T_n(2X)$  is the Chebyshev polynomial of the first kind. Then

$$\Gamma_i = T_{\bar{d}}(-(q^2 - q^{-2})K_i) \quad \text{for all } i \in \mathbb{Z}/3\mathbb{Z}$$

are central in  $U'_q(\mathfrak{so}_3)$ . A proof can be found in [11, Lemma 2].

On the other hand, while studying the quantum Teichmüller space, Checkhov and Fock [3, Example 2] were aware of a homomorphism  $\flat$  of  $U'_q(\mathfrak{so}_3)$  into the algebra generated by  $x_0^{\pm 1}$ ,  $x_1^{\pm 1}$ ,  $x_2^{\pm 1}$  subject to the relations

$$x_i x_i^{-1} = x_i^{-1} x_i = 1, \quad x_i x_{i+1} = q^2 x_{i+1} x_i \quad \text{for all } i \in \mathbb{Z}/3\mathbb{Z}.$$

The homomorphism  $\flat$  sends  $K_i$  to

$$\frac{q^{-1}x_i^{-1}x_{i+1}^{-1} + qx_i^{-1}x_{i+1} + q^{-1}x_i x_{i+1}}{q^2 - q^{-2}} \quad \text{for all } i \in \mathbb{Z}/3\mathbb{Z}.$$

Presently, the map  $\flat$  was shown to be injective by Iorgov [16, Proposition 1]. The images of the Casimir element and  $\Gamma_0, \Gamma_1, \Gamma_2$  were calculated out. As a consequence the relation

$$(5) \quad \begin{aligned} & 2(\lceil \bar{d}/2 \rceil - \lfloor \bar{d}/2 \rfloor - 1)(\Gamma_0 + \Gamma_1 + \Gamma_2 + 4)((-1)^{\lfloor \bar{d}/2 \rfloor} T_{\lfloor \bar{d}/2 \rfloor}(\Pi) + 2) - T_{\bar{d}}(\Pi) \\ & = q^{\bar{d}} \Gamma_0 \Gamma_1 \Gamma_2 + \Gamma_0^2 + \Gamma_1^2 + \Gamma_2^2 + 4(\lceil \bar{d}/2 \rceil - \lfloor \bar{d}/2 \rfloor) - 6 \end{aligned}$$

was first discovered in [16, Proposition 2], where  $\Pi$  is a normalization of (3) obtained by multiplying  $(q^2 - q^{-2})^2$  followed by adding  $q^2 + q^{-2}$ . The center of  $U'_q(\mathfrak{so}_3)$  was conjectured by Iorgov in [16, Conjecture 1] to be the commutative algebra over  $\mathbb{C}$  generated by  $\Gamma_0, \Gamma_1, \Gamma_2, \Pi$  subject to the relation (5). Until recently, the  $\mathbb{N}$ -filtration structure of  $U'_q(\mathfrak{so}_3)$  was utilized to confirm that  $\Gamma_i$  for all  $i \in \mathbb{Z}/3\mathbb{Z}$  and  $\Pi$  generate  $Z(U'_q(\mathfrak{so}_3))$  by Havlíček and Pošta [12, Theorem 3.1].

As we shall see in this paper, the whole ideas can be generalized to the case of  $\Delta$  by a replacement of the role of  $\flat$  with an embedding of  $\Delta$  into the tensor product of  $U_q(\mathfrak{sl}_2)$  with the Laurent ring  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$  given by Terwilliger [34]. Furthermore, a presentation for  $Z(\Delta)$  is given as follows. For each  $n \in \mathbb{N}$  let

$$\begin{aligned} \phi_n(X_0, X_1, X_2; X) &= T_n(X)T_n(X_0) + T_n(X_1)T_n(X_2), \\ \psi_n(X_0, X_1, X_2; X) &= T_{2n}(X) + T_n(X_0)^2 + T_n(X_1)^2 + T_n(X_2)^2 + T_n(X)T_n(X_0)T_n(X_1)T_n(X_2). \end{aligned}$$

To each  $i \in \mathbb{Z}/3\mathbb{Z}$  we associate a  $\mathbb{C}[X]$ -algebra automorphism  $i$  of  $\mathbb{C}[X_0, X_1, X_2, X]$  with

$$X_j^i = X_{i+j} \quad \text{for all } j \in \mathbb{Z}/3\mathbb{Z}.$$

For each  $n \in \mathbb{N}$  there exist unique polynomials  $\Phi_n(X_0, X_1, X_2; X)$ ,  $\Psi_n(X_0, X_1, X_2; X)$  over  $\mathbb{Z}$  such that

$$\Phi_n(\phi_m^0, \phi_m^1, \phi_m^2; \psi_m) = \phi_{mn}, \quad \Psi_n(\phi_m^0, \phi_m^1, \phi_m^2; \psi_m) = \psi_{mn} \quad \text{for all } m \in \mathbb{N}.$$

For instance

$$\begin{aligned} \Phi_0(X_0, X_1, X_2; X) &= 8, \\ \Phi_1(X_0, X_1, X_2; X) &= X_0, \\ \Phi_2(X_0, X_1, X_2; X) &= X_0^2 - 2X + 4, \\ \Phi_3(X_0, X_1, X_2; X) &= X_0^3 - 3(X - 1)X_0 + 3X_1X_2 \end{aligned}$$

and

$$\begin{aligned} \Psi_0(X_0, X_1, X_2; X) &= 30, \\ \Psi_1(X_0, X_1, X_2; X) &= X, \\ \Psi_2(X_0, X_1, X_2; X) &= X^2 - 8X + 2(X_0^2 + X_1^2 + X_2^2 - X_0X_1X_2) + 10, \\ \Psi_3(X_0, X_1, X_2; X) &= X^3 - 3(X_0X_1X_2 + X_0^2 + X_1^2 + X_2^2 + 1)X \\ &\quad + 3(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2 - X_0X_1X_2) + 6(X_0^2 + X_1^2 + X_2^2). \end{aligned}$$

Let  $\Omega$  denote a normalization of (2) obtained by subtracting from  $q^2 + q^{-2}$ . Applying the techniques from algebraic number theory,  $Z(\Delta)$  is shown to be the commutative algebra over  $\mathbb{C}$  generated by  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\Omega$  subject to the relation

$$\begin{aligned} &q^d \Phi_d^0(\alpha, \beta, \gamma; \Omega) T_d(A) + q^d \Phi_d^1(\alpha, \beta, \gamma; \Omega) T_d(B) + q^d \Phi_d^2(\alpha, \beta, \gamma; \Omega) T_d(C) \\ &= q^d T_d(A) T_d(B) T_d(C) + T_d(A)^2 + T_d(B)^2 + T_d(C)^2 + \Psi_d(\alpha, \beta, \gamma; \Omega) - 2. \end{aligned}$$

In one year after the Askey–Wilson algebra was formally proposed, the double affine Hecke algebra (DAHA) associated with a reduced affine root system was introduced by Cherednik as a tool to prove several conjectures made by Macdonald [4–6]. Later, the DAHA was extended by Sahi [30, §3] to any nonreduced affine root system. Consider the most general DAHA  $\mathfrak{H}$  of rank 1, namely of type  $(C_1^\vee, C_1)$ . Simply speaking, the algebra  $\mathfrak{H}$  is isomorphic to an algebra over  $\mathbb{C}$  generated by  $t_0$ ,  $t_1$ ,  $t_0^\vee$ ,  $t_1^\vee$  subject to the relations

$$\begin{aligned} (t_0 - k_0)(t_0 - k_0^{-1}) &= 0, & (t_1 - k_1)(t_1 - k_1^{-1}) &= 0, \\ (t_0^\vee - k_0^\vee)(t_0^\vee - k_0^{\vee-1}) &= 0, & (t_1^\vee - k_1^\vee)(t_1^\vee - k_1^{\vee-1}) &= 0, \\ t_0^\vee t_0 t_1^\vee t_1 &= q^{-1}, \end{aligned}$$

where  $k_0$ ,  $k_1$ ,  $k_0^\vee$ ,  $k_1^\vee$  are arbitrary nonzero parameters. The relationships between the Askey–Wilson algebra and the DAHA  $\mathfrak{H}$  of type  $(C_1^\vee, C_1)$  were investigated by Ito, Koornwinder and Terwilliger [18, 22, 23, 35]. The results from [22, §6] imply that there exists a homomorphism  $\Delta \rightarrow \mathfrak{H}$  that sends

$$A \mapsto t_1 t_0^\vee + (t_1 t_0^\vee)^{-1}, \quad B \mapsto t_1^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad C \mapsto t_0 t_1 + (t_0 t_1)^{-1}.$$

For  $q = 1$ , a presentation for  $Z(\mathfrak{H})$  was given by Oblomkov [28, Theorem 3.1]. For  $q$  a root of unity, a generalized but indefinite presentation for  $Z(\mathfrak{H})$  was subsequently obtained in [8, Theorem 6.12].

In the final section, the study of  $Z(\Delta)$  is used to explicitly extend the Oblomkov presentation for  $Z(\mathfrak{H})$  at roots of unity as below. Let  $\phi^0, \phi^1, \phi^2, \psi \in \mathbb{C}$  given by

$$\begin{aligned}\phi^0 &= (k_1^{\bar{d}} + k_1^{-\bar{d}})(k_0^{\vee\bar{d}} + k_0^{\vee-\bar{d}}) + q^{\bar{d}}(k_1^{\vee\bar{d}} + k_1^{\vee-\bar{d}})(k_0^{\bar{d}} + k_0^{-\bar{d}}), \\ \phi^1 &= (k_1^{\bar{d}} + k_1^{-\bar{d}})(k_1^{\vee\bar{d}} + k_1^{\vee-\bar{d}}) + q^{\bar{d}}(k_0^{\bar{d}} + k_0^{-\bar{d}})(k_0^{\vee\bar{d}} + k_0^{\vee-\bar{d}}), \\ \phi^2 &= (k_1^{\bar{d}} + k_1^{-\bar{d}})(k_0^{\bar{d}} + k_0^{-\bar{d}}) + q^{\bar{d}}(k_0^{\vee\bar{d}} + k_0^{\vee-\bar{d}})(k_1^{\vee\bar{d}} + k_1^{\vee-\bar{d}}), \\ \psi &= k_0^{2\bar{d}} + k_1^{2\bar{d}} + k_0^{\vee 2\bar{d}} + k_1^{\vee 2\bar{d}} + k_0^{-2\bar{d}} + k_1^{-2\bar{d}} + k_0^{\vee-2\bar{d}} + k_1^{\vee-2\bar{d}} \\ &\quad + (k_0^{\bar{d}} + k_0^{-\bar{d}})(k_1^{\bar{d}} + k_1^{-\bar{d}})(k_0^{\vee\bar{d}} + k_0^{\vee-\bar{d}})(k_1^{\vee\bar{d}} + k_1^{\vee-\bar{d}}).\end{aligned}$$

The polynomial ring  $\mathbb{C}[X_0, X_1, X_2]$  modulo the ideal generated by

$$q^{\bar{d}}X_0X_1X_2 + X_0^2 + X_1^2 + X_2^2 - \phi^0X_0 - \phi^1X_1 - \phi^2X_2 + \psi + 4$$

is isomorphic to  $Z(\mathfrak{H})$  induced from the mapping

$$X_0 \mapsto (t_1 t_0^{\vee})^{\bar{d}} + (t_1 t_0^{\vee})^{-\bar{d}}, \quad X_1 \mapsto (t_1^{\vee} t_1)^{\bar{d}} + (t_1^{\vee} t_1)^{-\bar{d}}, \quad X_2 \mapsto (t_0 t_1)^{\bar{d}} + (t_0 t_1)^{-\bar{d}}.$$

In addition, this paper will be employed to study the finite-dimensional irreducible  $\Delta$ -modules at roots of unity in the future paper [15], which has a potential application to the Racah–Wigner coefficients of  $U_q(\mathfrak{sl}_2)$  at roots of unity [14]. For  $q$  not a root of unity, a classification of finite-dimensional irreducible  $\Delta$ -modules was finished in [13, Theorem 4.7].

## 2. PRELIMINARIES

From now on, the ground field is set to be an arbitrary field  $\mathbb{F}$  because the statements in this paper are valid for any ground field not only the complex number field  $\mathbb{C}$ . The parameter  $q$  is always assumed to be a primitive  $d^{\text{th}}$  root of unity. Set  $\bar{d} = d$  if  $d$  is odd and  $\bar{d} = d/2$  if  $d$  is even.

The purpose of §2.1 is to give a brief review of the background on  $\Delta$  including the Poincaré–Birkhoff–Witt basis of  $\Delta$  and the induced  $\mathbb{N}$ -filtration structure of  $\Delta$ . In §2.2 we display the embedding  $\Delta \rightarrow \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U_q(\mathfrak{sl}_2)$ . The Iorgov’s identity is displayed in §2.3 to evaluate  $T_{\bar{d}}(A)^{\natural}, T_{\bar{d}}(B)^{\natural}, T_{\bar{d}}(C)^{\natural}$ . Recall the Concini–Kac presentation for  $Z(U_q(\mathfrak{sl}_2))$  in §2.4 to derive the relation in  $Z(\Delta)$  claimed in Introduction.

**2.1. The universal Askey–Wilson algebra  $\Delta$ .** To drop the assumption  $q^4 \neq 1$ , the universal Askey–Wilson algebra  $\Delta$  is slightly changed to be an  $\mathbb{F}$ -algebra with generators  $A, B, C, \alpha, \beta, \gamma$  and the relations assert that each of  $\alpha, \beta, \gamma$  is central in  $\Delta$  and

$$(6) \quad CB = q^2 BC + q(q^2 - q^{-2})A - q(q - q^{-1})\alpha,$$

$$(7) \quad CA = q^{-2} AC - q^{-1}(q^2 - q^{-2})B + q^{-1}(q - q^{-1})\beta,$$

$$(8) \quad BA = q^2 AB + q(q^2 - q^{-2})C - q(q - q^{-1})\gamma.$$

The Poincaré–Birkhoff–Witt theorem for  $\Delta$  was given in [33, Theorem 4.1]:

**Lemma 2.1.** *The monomials*

$$A^{i_0} B^{i_1} C^{i_2} \alpha^{j_0} \beta^{j_1} \gamma^{j_2} \quad \text{for all } i_0, i_1, i_2, j_0, j_1, j_2 \in \mathbb{N}$$

form an  $\mathbb{F}$ -basis of  $\Delta$ .

For any two submodules  $V, W$  of an algebra  $\mathcal{A}$ , denote by  $V \cdot W$  the submodule of  $\mathcal{A}$  spanned by  $vw$  for all  $v \in V$  and  $w \in W$ . For each  $n \in \mathbb{N}$  let  $\Delta_n$  denote the  $\mathbb{F}$ -subspace of  $\Delta$  spanned by

$$A^{i_0} B^{i_1} C^{i_2} \alpha^{j_0} \beta^{j_1} \gamma^{j_2} \quad \text{for all } i_0, i_1, i_2, j_0, j_1, j_2 \in \mathbb{N} \quad \text{with } i_0 + i_1 + i_2 + j_0 + j_1 + j_2 \leq n.$$

Recall from [33, §5] that the  $\mathbb{F}$ -subspaces  $\{\Delta_n\}_{n \in \mathbb{N}}$  of  $\Delta$  satisfy

$$(F1) \quad \Delta = \bigcup_{n \in \mathbb{N}} \Delta_n;$$

$$(F2) \quad \Delta_m \cdot \Delta_n \subseteq \Delta_{m+n} \quad \text{for all } m, n \in \mathbb{N}.$$

In other words, the increasing sequence

$$\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_n \subseteq \cdots$$

gives an  $\mathbb{N}$ -filtration of  $\Delta$ .

Recall from Introduction that the notation  $\Omega$  stands for the normalized Casimir element

$$(9) \quad q^2 + q^{-2} - qABC - q^2A^2 - q^{-2}B^2 - q^2C^2 + qA\alpha + q^{-1}B\beta + qC\gamma.$$

By [33, Proposition 7.4] we have

**Lemma 2.2.** *For each  $n \in \mathbb{N}$  the monomials*

$$A^{i_0} B^{i_1} C^{i_2} \alpha^{j_0} \beta^{j_1} \gamma^{j_2} \Omega^\ell \quad \begin{array}{l} i_0, i_1, i_2, j_0, j_1, j_2, \ell \in \mathbb{N}, \quad i_0 i_1 i_2 = 0, \\ i_0 + i_1 + i_2 + j_0 + j_1 + j_2 + 3\ell \leq n \end{array}$$

*form an  $\mathbb{F}$ -basis of  $\Delta_n$ .*

By the symmetry of the defining relations of  $\Delta$  there is an  $\mathbb{F}$ -algebra automorphism  $\rho$  of  $\Delta$  that sends

$$(A, B, C, \alpha, \beta, \gamma) \mapsto (B, C, A, \beta, \gamma, \alpha).$$

Moreover [33, Theorem 6.4] showed that

**Lemma 2.3.**  $\Omega^\rho = \Omega$ .

**2.2. An embedding of  $\Delta$  into  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U_q(\mathfrak{sl}_2)$ .** Assuming that  $q^2 \neq 1$  the quantum group  $U_q(\mathfrak{sl}_2)$  is an  $\mathbb{F}$ -algebra generated by  $e, k^{\pm 1}, f$  subject to the relations

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1, \\ ke &= q^2ek, \quad kf = q^{-2}fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

Let us abbreviate  $U = U_q(\mathfrak{sl}_2)$ . The generators  $e, k^{\pm 1}, f$  are called the *Chevalley generators* of  $U$ . The Casimir element of  $U$  is defined as

$$ef + \frac{q^{-1}k + qk^{-1}}{(q - q^{-1})^2}.$$

Here we denote by  $\Lambda$  a normalized Casimir element of  $U$  obtained by multiplying the factor  $(q - q^{-1})^2$ . The Poincaré–Birkhoff–Witt theorem for  $U$  can be found in [20, Theorem 1.5]:

**Lemma 2.4.** *The monomials*

$$f^s k^{\pm i} e^r \quad \text{for all } i, r, s \in \mathbb{N}$$

*form an  $\mathbb{F}$ -basis of  $U$ .*

For each  $n \in \mathbb{Z}$  let  $U_n$  denote the  $\mathbb{F}$ -subspace of  $U$  spanned by

$$f^s k^{\pm i} e^r \quad \text{for all } i, r, s \in \mathbb{N} \text{ with } r - s = n.$$

Recall from [20, §1.9] that the  $\mathbb{F}$ -subspaces  $\{U_n\}_{n \in \mathbb{Z}}$  of  $U$  satisfy

$$(G1) \quad U = \bigoplus_{n \in \mathbb{Z}} U_n;$$

$$(G2) \quad U_m \cdot U_n \subseteq U_{m+n} \text{ for all } m, n \in \mathbb{Z}.$$

In other words, the  $\mathbb{F}$ -spaces  $\{U_n\}_{n \in \mathbb{Z}}$  give a  $\mathbb{Z}$ -gradation of  $U$ . By (G1) any element  $u \in U$  can be uniquely written as

$$\sum_{n \in \mathbb{Z}} u_n$$

where  $u_n \in U_n$  and almost all zero. For each  $n \in \mathbb{Z}$  the element  $u_n$  is called the *homogeneous component of  $u$  of degree  $n$*  or simply the  *$n$ -homogeneous component of  $u$* .

Consider the elements

$$x = k^{-1} - q^{-1}(q - q^{-1})ek^{-1}, \quad y^{\pm 1} = k^{\pm 1}, \quad z = k^{-1} + (q - q^{-1})f.$$

Solving for  $e$ ,  $k^{\pm 1}$ ,  $f$  it yields that

$$e = \frac{q(1 - xy)}{q - q^{-1}}, \quad k^{\pm 1} = y^{\pm 1}, \quad f = \frac{z - y^{-1}}{q - q^{-1}}.$$

Thus  $x$ ,  $y^{\pm 1}$ ,  $z$  generate  $U$  and they are called the *equitable generators* of  $U$  [19, Definition 2.2]. Let  $a$ ,  $b$ ,  $c$  denote three commuting indeterminates over  $\mathbb{F}$ . Extending the ground field  $\mathbb{F}$  of  $U$  to the Laurent polynomial ring  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ , a recent work [34, Theorem 2.16–2.18] of Terwilliger showed that

**Lemma 2.5.** *There exists a unique  $\mathbb{F}$ -algebra injective homomorphism  $\mathfrak{h} : \Delta \rightarrow \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U$  with*

$$\begin{aligned} A^{\mathfrak{h}} &= ax + a^{-1}y + qbc^{-1}(1 - xy), \\ B^{\mathfrak{h}} &= by + b^{-1}z + qca^{-1}(1 - yz), \\ C^{\mathfrak{h}} &= cz + c^{-1}x + qab^{-1}(1 - zx), \\ \alpha^{\mathfrak{h}} &= (b + b^{-1})(c + c^{-1}) + (a + a^{-1})\Lambda, \\ \beta^{\mathfrak{h}} &= (c + c^{-1})(a + a^{-1}) + (b + b^{-1})\Lambda, \\ \gamma^{\mathfrak{h}} &= (a + a^{-1})(b + b^{-1}) + (c + c^{-1})\Lambda, \\ \Omega^{\mathfrak{h}} &= (a + a^{-1})^2 + (b + b^{-1})^2 + (c + c^{-1})^2 + (a + a^{-1})(b + b^{-1})(c + c^{-1})\Lambda + \Lambda^2 - 2. \end{aligned}$$

Let  $U'$  denote the  $\mathbb{F}$ -subalgebra of  $U$  generated by  $x$ ,  $y$ ,  $z$ . In terms of the equitable generators

$$\Lambda = qx + q^{-1}y + qz - qxyz.$$

Hence  $\Lambda \in U'$ . By Lemma 2.5 the image of  $\mathfrak{h}$  is contained in  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U'$ . More precisely,

**Lemma 2.6.**  $\mathbb{F}[\alpha, \beta, \gamma, \Omega]^{\mathfrak{h}}$  is contained in  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}[\Lambda]$ .

In [34, Lemma 10.11 and Proposition 10.14] the  $\mathbb{F}$ -algebra automorphism  $\rho : \Delta \rightarrow \Delta$  below Lemma 2.2 is extended as follows.

**Lemma 2.7.** *There exists a unique  $\mathbb{F}$ -algebra automorphism  $\tilde{\rho}$  of  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U'$  that sends*

$$(a, b, c, x, y, z) \mapsto (b, c, a, y, z, x).$$

Moreover the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathfrak{h}} & \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U' \\ \rho \downarrow & & \downarrow \tilde{\rho} \\ \Delta & \xrightarrow{\mathfrak{h}} & \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U' \end{array}$$

commutes.

By [34, Lemma 10.13] we have

**Lemma 2.8.**  $\Lambda^{\tilde{\rho}} = \Lambda$ .

**2.3. Gaussian binomial coefficients and Chebyshev polynomials.** The main purpose of this subsection is to display two identities to facilitate the evaluation of  $T_d(A)^{\natural}$ ,  $T_d(B)^{\natural}$ ,  $T_d(C)^{\natural}$ . Let  $Q$  denote an indeterminate over the rational number field. Recall the notation

$$[n] = \frac{Q^n - Q^{-n}}{Q - Q^{-1}} \quad \text{for all } n \in \mathbb{Z}$$

and the Gaussian binomial coefficients

$$\begin{bmatrix} n \\ i \end{bmatrix} = \prod_{j=1}^i \frac{[n-j+1]}{[j]} \quad \text{for all } i \in \mathbb{N} \text{ and } n \in \mathbb{Z}.$$

The Gaussian binomial coefficients are shown to be contained in  $\mathbb{Z}[Q^{\pm 1}]$ . Since  $\lim_{Q \rightarrow 1} [n] = n$  for  $n \in \mathbb{Z}$  the binomial coefficients are a limit case of the Gaussian binomial coefficients. The binomial formula can be generalized as follows.

**Proposition 2.9.** *If two elements  $R, S$  in a  $\mathbb{Z}[Q^{\pm 1}]$ -algebra satisfy  $SR = Q^2RS$ , then*

$$(R + S)^n = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix} Q^{i(n-i)} R^{n-i} S^i \quad \text{for all } n \in \mathbb{N}.$$

*In particular, if  $Q$  is a primitive  $d^{\text{th}}$  root of unity then  $(R + S)^d = R^d + S^d$ .*

Set  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . By [21, §9.8.2] the normalized Chebyshev polynomials of the first kind  $\{T_n(X)\}_{n \in \mathbb{N}}$  given in (4) satisfy

$$XT_n(X) = T_{n+1}(X) + T_{n-1}(X) \quad \text{for } n \in \mathbb{N}^*$$

with  $T_0(X) = 2$  and  $T_1(X) = X$ .

**Lemma 2.10.** (i)  $T_n(X + X^{-1}) = X^n + X^{-n}$  for all  $n \in \mathbb{N}$ .

(ii)  $T_m(T_n(X)) = T_{mn}(X)$  for all  $m, n \in \mathbb{N}$ .

*Proof.* To see (i) proceed by a routine induction on  $n$  and apply the above recurrence relation. To see (ii) replace  $X$  by  $X + X^{-1}$  in  $T_m(T_n(X))$  and apply (i).  $\square$

Recall the central elements  $\Gamma_0, \Gamma_1, \Gamma_2$  of  $U'_q(\mathfrak{so}_3)$  and the embedding  $\flat$  of  $U'_q(\mathfrak{so}_3)$  from Introduction. To evaluate the images of  $\Gamma_0, \Gamma_1, \Gamma_2$  under  $\flat$  Iorgov established the identity [16, Lemma 1]:

**Proposition 2.11.** *If three elements  $R^{\pm 1}, S$  in a  $\mathbb{Z}[Q^{\pm 1}]$ -algebra satisfy the relations  $RR^{-1} = R^{-1}R = 1$  and  $SR = Q^2RS$ , then  $T_n(R + S + R^{-1})$  is equal to*

$$R^n + R^{-n} + \sum_{i=1}^n \sum_{j=0}^{n-i} \frac{[n]}{[i]} \begin{bmatrix} i+j-1 \\ i-1 \end{bmatrix} \begin{bmatrix} n-j-1 \\ i-1 \end{bmatrix} Q^{i(n-i-2j)} R^{n-i-2j} S^i$$

*for each  $n \in \mathbb{N}$ . In particular, if  $Q$  is a primitive  $d^{\text{th}}$  root of unity then  $T_d(R + S + R^{-1}) = R^d + S^d + R^{-d}$ .*

Henceforth each element in  $\mathbb{Z}[Q^{\pm 1}]$  is also treated as an element in  $\mathbb{F}$  via the ring homomorphism  $\mathbb{Z}[Q^{\pm 1}] \rightarrow \mathbb{F}$  given by  $1 \mapsto 1$  and  $Q \mapsto q$ .

**2.4. A presentation for  $Z(U)$ .** In [7, Theorem 4.2] Concini and Kac showed that  $Z(U)$  is the commutative  $\mathbb{F}$ -algebra generated by  $e^d, k^{\pm d}, f^d, \Lambda$  subject to the relations  $k^d k^{-d} = 1$  and

$$(10) \quad T_d(\Lambda) = (q - q^{-1})^{2d} e^d f^d + q^d (k^d + k^{-d}).$$

Note that  $d$  could be any positive integer except 1 because  $q^2 \neq 1$  is necessary in the definition of  $U$ .

**Lemma 2.12.** *For the tower  $\mathbb{F} \subseteq \mathbb{F}[\Lambda] \subseteq Z(U)$  the following hold:*

(i) *The elements*

$$\Lambda^n \quad \text{for all } n \in \mathbb{N}$$

*form an  $\mathbb{F}$ -basis of  $\mathbb{F}[\Lambda]$ .*

(ii) *The elements*

$$k^{\pm di} f^{\bar{d}j}, \quad k^{\pm di}, \quad k^{\pm di} e^{\bar{d}j} \quad \text{for all } i \in \mathbb{N} \text{ and } j \in \mathbb{N}^*$$

*form an  $\mathbb{F}[\Lambda]$ -basis of  $Z(U)$ .*

*Proof.* Clearly  $\mathbb{F}[\Lambda]$  is spanned by  $\{\Lambda^n\}_{n \in \mathbb{N}}$  over  $\mathbb{F}$ . An induction on  $n \in \mathbb{N}$  yields that

$$(11) \quad \Lambda^n = (q - q^{-1})^{2n} f^n e^n + \sum_{i=0}^{n-1} f^i k_i e^i$$

where  $k_i \in \mathbb{F}[k^{\pm 1}]$ . By Lemma 2.4 the elements  $\{\Lambda^n\}_{n \in \mathbb{N}}$  are linearly independent over  $\mathbb{F}$ . This shows (i).

Relation (11) along with Lemma 2.4 also implies that the elements

$$\Lambda^n k^{\pm di} f^{\bar{d}j}, \quad \Lambda^n k^{\pm di}, \quad \Lambda^n k^{\pm di} e^{\bar{d}j} \quad \text{for all } i, n \in \mathbb{N} \text{ and } j \in \mathbb{N}^*$$

are linearly independent over  $\mathbb{F}$ . It follows that  $k^{\pm di} f^{\bar{d}j}$ ,  $k^{\pm di}$ ,  $k^{\pm di} e^{\bar{d}j}$  for all  $i \in \mathbb{N}$  and  $j \in \mathbb{N}^*$  are linearly independent over  $\mathbb{F}[\Lambda]$ . Since  $Z(U)$  is generated by  $e^{\bar{d}}$ ,  $k^{\pm \bar{d}}$ ,  $f^{\bar{d}}$ ,  $\Lambda$  the monomials

$$f^{\bar{d}r} k^{\pm di} e^{\bar{d}s} \quad \text{for all } i, r, s \in \mathbb{N}$$

span  $Z(U)$  over  $\mathbb{F}[\Lambda]$ . Applying (10) each of these monomials is an  $\mathbb{F}[\Lambda]$ -linear combination of  $k^{\pm di} f^{\bar{d}j}$ ,  $k^{\pm di}$ ,  $k^{\pm di} e^{\bar{d}j}$  for all  $i \in \mathbb{N}$  and  $j \in \mathbb{N}^*$ . This shows (ii).  $\square$

### 3. THREE CENTRAL ELEMENTS OF $\Delta$ AT ROOTS OF UNITY

**3.1. The elements  $T_{\bar{d}}(A)$ ,  $T_{\bar{d}}(B)$ ,  $T_{\bar{d}}(C)$  central in  $\Delta$ .** Observe that the number  $\bar{d}$  is chosen to be the order of the root of unity  $q^2$ . Let  $Y$  denote an indeterminate over  $\mathbb{F}$  commuting with  $X$ . Consider the Laurent polynomials

$$\Theta_i(Y) = q^{2i} Y^{-1} + q^{-2i} Y \quad \text{for all } i \in \mathbb{Z}.$$

**Lemma 3.1.** *For all  $i, j \in \mathbb{Z}$*

- (i)  $\Theta_i(Y) = \Theta_j(Y)$  *if and only if  $\bar{d}$  divides  $i - j$ ;*
- (ii)  $T_{\bar{d}}(\Theta_i(Y)) = Y^{\bar{d}} + Y^{-\bar{d}}$ .

*Proof.* (i) follows by the factorization  $\Theta_i(Y) - \Theta_j(Y) = (q^{i-j} - q^{j-i})(q^{i+j} Y^{-1} - q^{-i-j} Y)$ . To see (ii) apply Lemma 2.10(i).  $\square$

By virtue of Lemma 3.1 one may prove that

**Theorem 3.2.** *The elements  $T_{\bar{d}}(A)$ ,  $T_{\bar{d}}(B)$ ,  $T_{\bar{d}}(C)$  are central in  $\Delta$ .*

*Proof.* Without loss we show that  $T_{\bar{d}}(A)$  is central in  $\Delta$  since the  $\mathbb{F}$ -algebra automorphism  $\rho : \Delta \rightarrow \Delta$  cyclically permutes  $A, B, C$ . The  $\mathbb{F}$ -algebra  $\Delta$  is generated by  $A, B, C$  and three central elements  $\alpha, \beta, \gamma$ . Evidently  $T_{\bar{d}}(A)$  commutes with  $A$ . Thus it suffices to show that  $T_{\bar{d}}(A)$  commutes with  $B$  and  $C$ . To evaluate  $B T_{\bar{d}}(A)$  we begin to express  $B A^n$  ( $n \in \mathbb{N}$ ) as a linear combination of the  $\mathbb{F}$ -basis of  $\Delta$  given in Lemma 2.1:



**Lemma 3.3.** For each  $n \in \mathbb{N}$

$$BA^n = P_n(A)B + Q_n(A)C + R_n(A)\beta + S_n(A)\gamma$$

where  $P_n(X)$ ,  $Q_n(X)$ ,  $R_n(X)$ ,  $S_n(X) \in \mathbb{F}[X]$  are recurrently defined by  $P_0(X) = 1$ ,  $Q_0(X) = 0$ ,  $R_0(X) = 0$ ,  $S_0(X) = 0$  and for  $n \in \mathbb{N}^*$

$$(12) \quad P_n(X) = q^2 X P_{n-1}(X) - q^{-1}(q^2 - q^{-2})Q_{n-1}(X),$$

$$(13) \quad Q_n(X) = q^{-2} X Q_{n-1}(X) + q(q^2 - q^{-2})P_{n-1}(X),$$

$$(14) \quad R_n(X) = X R_{n-1}(X) + q^{-1}(q - q^{-1})Q_{n-1}(X),$$

$$(15) \quad S_n(X) = X S_{n-1}(X) - q(q - q^{-1})P_{n-1}(X).$$

*Proof.* Proceed by induction on  $n$ . There is nothing to prove for  $n = 0$ . By induction hypothesis

$$(16) \quad BA^n = P_{n-1}(A)BA + Q_{n-1}(A)CA + R_{n-1}(A)A\beta + S_{n-1}(A)A\gamma \quad \text{for } n \geq 1.$$

This lemma follows by substituting (7), (8) into (16).  $\square$

Applying Lemma 3.3 a direct calculation yields that

$$\begin{aligned} P_1(X) &= q^2 X, & Q_1(X) &= q(q^2 - q^{-2}), & R_1(X) &= 0, & S_1(X) &= -q(q - q^{-1}), \\ P_2(X) &= q^4 X^2 - (q^2 - q^{-2})^2, & Q_2(X) &= q(q^4 - q^{-4})X, \\ R_2(X) &= (q - q^{-1})(q^2 - q^{-2}), & S_2(X) &= -q^2(q^2 - q^{-2})X. \end{aligned}$$

In particular, if  $q^2 = 1$  then  $P_1(X) = X$ ,  $Q_1(X) = 0$ ,  $R_1(X) = 0$ ,  $S_1(X) = 0$  and if  $q^4 = 1$  then  $P_2(X) = X^2$ ,  $Q_2(X) = 0$ ,  $R_2(X) = 0$ ,  $S_2(X) = 0$ . Thus this theorem holds for  $d = 1, 2$ .

For the rest assume that  $d > 2$ . It seems difficult to find the closed forms of  $P_n(X)$ ,  $Q_n(X)$ ,  $R_n(X)$ ,  $S_n(X)$  for any  $n \in \mathbb{N}$ . Instead we evaluate these polynomials at  $\{\Theta_i(Y)\}_{i \in \mathbb{Z}}$ .

**Lemma 3.4.** For all  $i \in \mathbb{Z}$  and  $n \in \mathbb{N}$

$$\begin{aligned} P_n(\Theta_i(Y)) &= \frac{\Theta_{i+1}(Y) - q^2 \Theta_i(Y)}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)} \Theta_{i-1}(Y)^n + \frac{q^2 \Theta_i(Y) - \Theta_{i-1}(Y)}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)} \Theta_{i+1}(Y)^n, \\ Q_n(\Theta_i(Y)) &= q(q^2 - q^{-2}) \frac{\Theta_{i+1}(Y)^n - \Theta_{i-1}(Y)^n}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)}, \\ R_n(\Theta_i(Y)) &= \frac{(q - q^{-1})(q^2 - q^{-2})}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)} \left( \frac{\Theta_{i+1}(Y)^n - \Theta_i(Y)^n}{\Theta_{i+1}(Y) - \Theta_i(Y)} - \frac{\Theta_i(Y)^n - \Theta_{i-1}(Y)^n}{\Theta_i(Y) - \Theta_{i-1}(Y)} \right), \\ S_n(\Theta_i(Y)) &= \frac{q(q - q^{-1})}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)} \left( \frac{\Theta_{i+1}(Y)^n - \Theta_i(Y)^n}{\Theta_{i+1}(Y) - \Theta_i(Y)} (\Theta_{i-1}(Y) - q^2 \Theta_i(Y)) \right. \\ &\quad \left. - \frac{\Theta_i(Y)^n - \Theta_{i-1}(Y)^n}{\Theta_i(Y) - \Theta_{i-1}(Y)} (\Theta_{i+1}(Y) - q^2 \Theta_i(Y)) \right). \end{aligned}$$

*Proof.* Multiply (12), (13) by  $q^{-2}X$ ,  $-q^{-1}(q^2 - q^{-2})$  respectively. The difference of the resulting equations gives

$$q^{-2} X P_n(X) + q^{-1}(q^2 - q^{-2})Q_n(X) = (X^2 + (q^2 - q^{-2})^2)P_{n-1}(X) \quad \text{for } n \in \mathbb{N}^*.$$

Subtracting the above from the equation obtained by replacing the index  $n$  in (12) by  $n + 1$ , it follows that the polynomials  $\{P_n(X)\}_{n \in \mathbb{N}}$  satisfy

$$P_{n+1}(X) = (q^2 + q^{-2})X P_n(X) - (X^2 + (q^2 - q^{-2})^2)P_{n-1}(X) \quad \text{for } n \in \mathbb{N}^*.$$

Fix  $i \in \mathbb{Z}$ . Substituting  $X = \Theta_i(Y)$  into the above relation, the characteristic polynomial of the resulting recurrence relation is

$$K(X) = X^2 - (q^2 + q^{-2})\Theta_i(Y)X + \Theta_i(Y)^2 + (q^2 - q^{-2})^2.$$

It is straightforward to verify that  $\Theta_{i-1}(Y)$  and  $\Theta_{i+1}(Y)$  are two roots of  $K(X)$ . Under the assumption  $\bar{d} > 2$  the Laurent polynomials  $\Theta_{i-1}(Y)$ ,  $\Theta_{i+1}(Y)$  are distinct by Lemma 3.1(i). Therefore  $P_n(\Theta_i(Y))$  is of the form

$$\Xi(Y)\Theta_{i-1}(Y)^n + \Sigma(Y)\Theta_{i+1}(Y)^n \quad \text{for all } n \in \mathbb{N},$$

where  $\Xi(Y)$ ,  $\Sigma(Y) \in \mathbb{F}(Y)$ . The functions  $\Xi(Y)$ ,  $\Sigma(Y)$  can be solved from the matrix equation

$$\begin{pmatrix} 1 & 1 \\ \Theta_{i-1}(Y) & \Theta_{i+1}(Y) \end{pmatrix} \begin{pmatrix} \Xi(Y) \\ \Sigma(Y) \end{pmatrix} = \begin{pmatrix} 1 \\ q^2\Theta_i(Y) \end{pmatrix},$$

which obtains from  $P_0(X) = 1$  and  $P_1(X) = q^2X$ . The identity for  $Q_n(\Theta_i(Y))$  follows by a similar argument.

Proceed by induction on  $n$  to verify the identities for  $R_n(\Theta_i(Y))$  and  $S_n(\Theta_i(Y))$ . It holds for  $n = 0$  since  $R_0(X) = 0$  and  $S_0(X) = 0$ . For  $n \geq 1$  apply induction hypotheses to (14), (15) with  $X = \Theta_i(Y)$  and simplify them by using the identities for  $Q_{n-1}(\Theta_i(Y))$ ,  $P_{n-1}(\Theta_i(Y))$  respectively.  $\square$

As a consequence of Lemma 3.3 the element  $BT_n(A)$  is equal to

$$\mathcal{P}_n(A)B + \mathcal{Q}_n(A)C + \mathcal{R}_n(A)\beta + \mathcal{S}_n(A)\gamma \quad \text{for each } n \in \mathbb{N},$$

where  $\mathcal{P}_n(X)$ ,  $\mathcal{Q}_n(X)$ ,  $\mathcal{R}_n(X)$ ,  $\mathcal{S}_n(X) \in \mathbb{F}[X]$  are obtained from (4) by replacing  $X^{n-2i}$  by  $P_{n-2i}(X)$ ,  $Q_{n-2i}(X)$ ,  $R_{n-2i}(X)$ ,  $S_{n-2i}(X)$  for all  $0 \leq i \leq \lfloor n/2 \rfloor$ , respectively. To see that  $T_{\bar{d}}(A)$  commutes with  $B$  it needs to show that

$$\mathcal{P}_{\bar{d}}(X) = T_{\bar{d}}(X), \quad \mathcal{Q}_{\bar{d}}(X) = \mathcal{R}_{\bar{d}}(X) = \mathcal{S}_{\bar{d}}(X) = 0.$$

By Lemma 3.3 a routine induction yields that  $P_n(X)$  is of degree  $n$  with leading coefficient  $q^{2n}$  and hence so is  $\mathcal{P}_n(X)$ . In particular  $\mathcal{P}_{\bar{d}}(X)$  is a monic polynomial of degree  $\bar{d}$  as well as  $T_{\bar{d}}(X)$ . Applying the formula of  $P_n(\Theta_i(Y))$  in Lemma 3.4 a direct calculation yields that

$$\mathcal{P}_n(\Theta_i(Y)) = \frac{\Theta_{i+1}(Y) - q^2\Theta_i(Y)}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)} T_n(\Theta_{i-1}(Y)) + \frac{q^2\Theta_i(Y) - \Theta_{i-1}(Y)}{\Theta_{i+1}(Y) - \Theta_{i-1}(Y)} T_n(\Theta_{i+1}(Y))$$

for all  $i \in \mathbb{Z}$ . Simplifying the above equality by using Lemma 3.1(ii) it follows that

$$\mathcal{P}_{\bar{d}}(\Theta_i(Y)) = Y^{\bar{d}} + Y^{-\bar{d}} \quad \text{for all } i \in \mathbb{Z}.$$

So far we have seen that  $\mathcal{P}_{\bar{d}}(X)$  and  $T_{\bar{d}}(X)$  are monic polynomials of degree  $\bar{d}$  and agree on  $\{\Theta_i(Y)\}_{i=0}^{\bar{d}-1}$  which are pairwise distinct by Lemma 3.1(i). Therefore  $\mathcal{P}_{\bar{d}}(X) = T_{\bar{d}}(X)$  by division algorithm. A similar argument shows that each of  $\mathcal{Q}_{\bar{d}}(X)$ ,  $\mathcal{R}_{\bar{d}}(X)$ ,  $\mathcal{S}_{\bar{d}}(X)$  is identically zero. By a similar argument  $T_{\bar{d}}(A)$  commutes with  $C$ . The result follows.  $\square$

As consequences of Theorem 3.2 the following elements are contained in  $Z(\Delta)$ .

**Proposition 3.5.** (i)  $T_n(A)$ ,  $T_n(B)$ ,  $T_n(C)$  are central in  $\Delta$  for all nonnegative multiples  $n$  of  $\bar{d}$ .  
(ii)  $\prod_{i=0}^{\bar{d}-1} (A - \Theta_i(\lambda))$ ,  $\prod_{i=0}^{\bar{d}-1} (B - \Theta_i(\lambda))$ ,  $\prod_{i=0}^{\bar{d}-1} (C - \Theta_i(\lambda))$  are central in  $\Delta$  for all nonzero  $\lambda \in \mathbb{F}$ .

*Proof.* (i) is immediate from Lemma 2.10(ii). By the division algorithm Lemma 3.1(i) implies that

$$(17) \quad T_{\bar{d}}(X) = \prod_{i=0}^{\bar{d}-1} (X - \Theta_i(Y)) + Y^{\bar{d}} + Y^{-\bar{d}}.$$

(ii) is immediate from (17).  $\square$

Equation (17) also implies that the zeros of  $T_{\bar{d}}(X)$  are  $\{\Theta_i(\lambda)\}_{i=0}^{\bar{d}-1}$  where  $\lambda$  is a root of  $Y^{\bar{d}} + Y^{-\bar{d}}$ . More generally, please refer to [32] for zeros of the Askey-Wilson polynomials at roots of unity.

**3.2. The images of  $T_{\vec{d}}(A)$ ,  $T_{\vec{d}}(B)$ ,  $T_{\vec{d}}(C)$  under  $\natural$ .** Throughout this subsection assume that  $q^2 \neq 1$ . Recall the embedding  $\natural : \Delta \rightarrow \mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U$  from Lemma 2.5. This subsection is devoted to computing  $T_{\vec{d}}(A)^{\natural}$ ,  $T_{\vec{d}}(B)^{\natural}$ ,  $T_{\vec{d}}(C)^{\natural}$ . To do this we need the following conversion formulae between  $e^{\vec{d}}$ ,  $k^{\pm \vec{d}}$ ,  $f^{\vec{d}}$  and  $x^{\vec{d}}$ ,  $y^{\pm \vec{d}}$ ,  $z^{\vec{d}}$ .

**Lemma 3.6.** *Assume that  $q^2 \neq 1$ . Then the following hold:*

(i) *The elements  $x^{\vec{d}}$ ,  $y^{\pm \vec{d}}$ ,  $z^{\vec{d}}$  are equal to*

$$k^{-\vec{d}} - q^{\vec{d}}(q - q^{-1})^{\vec{d}} e^{\vec{d}} k^{-\vec{d}}, \quad k^{\pm \vec{d}}, \quad k^{-\vec{d}} + (q - q^{-1})^{\vec{d}} f^{\vec{d}}$$

*respectively. In particular  $x^{\vec{d}}$ ,  $y^{\pm \vec{d}}$ ,  $z^{\vec{d}}$  are central in  $U$ .*

(ii) *The elements  $e^{\vec{d}}$ ,  $k^{\pm \vec{d}}$ ,  $f^{\vec{d}}$  are equal to*

$$q^{\vec{d}}(q - q^{-1})^{-\vec{d}}(1 - x^{\vec{d}} y^{\vec{d}}), \quad y^{\pm \vec{d}}, \quad (q - q^{-1})^{-\vec{d}}(z^{\vec{d}} - y^{-\vec{d}})$$

*respectively.*

*Proof.* To get (ii) solve (i) for  $e^{\vec{d}}$ ,  $k^{\pm \vec{d}}$ ,  $f^{\vec{d}}$ . Thus it only needs to prove (i). Since  $y^{\pm 1} = k^{\pm 1}$  the equality  $y^{\pm \vec{d}} = k^{\pm \vec{d}}$  holds. Recall that

$$x = k^{-1} - q^{-1}(q - q^{-1})ek^{-1}.$$

The first two defining relations of  $U$  imply that

$$(18) \quad ek^{-1} = q^2 k^{-1} e.$$

Applying Proposition 2.9 with  $(R, S) = (k^{-1}, -q^{-1}(q - q^{-1})ek^{-1})$  it follows that

$$x^{\vec{d}} = k^{-\vec{d}} + (-1)^{\vec{d}} q^{-\vec{d}}(q - q^{-1})^{\vec{d}}(ek^{-1})^{\vec{d}}.$$

To further simplify the addend in the above identity, one may use the following lemmas:

**Lemma 3.7.**  $(ek^{-1})^n = q^{n(n-1)} e^n k^{-n}$  for all  $n \in \mathbb{N}$ .

*Proof.* This lemma follows by inductively applying (18). □

**Lemma 3.8.**  $q^{\vec{d}(\vec{d}-1)} = (-1)^{\vec{d}-1}$ .

*Proof.* By the setting of  $\vec{d}$  the left-hand side is equal to 1 (resp.  $-1$ ) if  $\vec{d}$  is odd (resp. even). □

By a similar argument the identity for  $z^{\vec{d}}$  follows. □

We are ready to evaluate the images of  $T_{\vec{d}}(A)$ ,  $T_{\vec{d}}(B)$ ,  $T_{\vec{d}}(C)$  under  $\natural$ .

**Theorem 3.9.** *Assume that  $q^2 \neq 1$ . Then  $T_{\vec{d}}(A)^{\natural}$ ,  $T_{\vec{d}}(B)^{\natural}$ ,  $T_{\vec{d}}(C)^{\natural}$  are central in  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U$ . Moreover  $T_{\vec{d}}(A)^{\natural}$ ,  $T_{\vec{d}}(B)^{\natural}$ ,  $T_{\vec{d}}(C)^{\natural}$  are equal to*

$$(19) \quad a^{\vec{d}} x^{\vec{d}} + a^{-\vec{d}} y^{\vec{d}} + q^{\vec{d}} b^{\vec{d}} c^{-\vec{d}}(1 - x^{\vec{d}} y^{\vec{d}}),$$

$$(20) \quad b^{\vec{d}} y^{\vec{d}} + b^{-\vec{d}} z^{\vec{d}} + q^{\vec{d}} c^{\vec{d}} a^{-\vec{d}}(1 - y^{\vec{d}} z^{\vec{d}}),$$

$$(21) \quad c^{\vec{d}} z^{\vec{d}} + c^{-\vec{d}} x^{\vec{d}} + q^{\vec{d}} a^{\vec{d}} b^{-\vec{d}}(1 - z^{\vec{d}} x^{\vec{d}}),$$

*respectively. Their nonzero homogeneous components are as follows:*

| degree    | $T_{\vec{d}}(A)^{\natural}$   | degree     | $T_{\vec{d}}(B)^{\natural}$   |
|-----------|---|------------|---|
| 0         | $a^{\vec{d}} k^{-\vec{d}} + a^{-\vec{d}} k^{\vec{d}}$   | $-\vec{d}$ | $(q - q^{-1})^{\vec{d}}(b^{-\vec{d}} - q^{\vec{d}} a^{-\vec{d}} c^{\vec{d}} k^{\vec{d}}) f^{\vec{d}}$ |
| $\vec{d}$ | $(q - q^{-1})^{\vec{d}}(b^{\vec{d}} c^{-\vec{d}} - q^{\vec{d}} a^{\vec{d}} k^{-\vec{d}}) e^{\vec{d}}$ | 0          | $b^{-\vec{d}} k^{-\vec{d}} + b^{\vec{d}} k^{\vec{d}}$   |

| degree     | $T_{\vec{d}}(C)^{\natural}$   |
|------------|---|
| $-\vec{d}$ | $(q - q^{-1})^{\vec{d}}(c^{\vec{d}} - q^{\vec{d}}a^{\vec{d}}b^{-\vec{d}}k^{-\vec{d}})f^{\vec{d}}$   |
| 0          | $-q^{\vec{d}}a^{\vec{d}}b^{-\vec{d}}k^{-2\vec{d}} + (c^{\vec{d}} + c^{-\vec{d}})k^{-\vec{d}} + a^{\vec{d}}b^{-\vec{d}}(q - q^{-1})^{2\vec{d}}f^{\vec{d}}k^{-\vec{d}}e^{\vec{d}} + q^{\vec{d}}a^{\vec{d}}b^{-\vec{d}}$ |
| $\vec{d}$  | $(q - q^{-1})^{\vec{d}}(a^{\vec{d}}b^{-\vec{d}}k^{-\vec{d}} - q^{\vec{d}}c^{-\vec{d}})k^{-\vec{d}}e^{\vec{d}}$  |

*Proof.* Express  $A^{\natural}$  given in Lemma 2.5 in terms of Chevalley generators:

$$A^{\natural} = ak^{-1} + a^{-1}k + (q - q^{-1})(bc^{-1}e - aq^{-1}ek^{-1}).$$

By relation (18), Proposition 2.11 is applicable to  $(R, S) = (ak^{-1}, (q - q^{-1})(bc^{-1}e - aq^{-1}ek^{-1}))$ . It follows that

$$T_{\vec{d}}(A)^{\natural} = a^{\vec{d}}k^{-\vec{d}} + a^{-\vec{d}}k^{\vec{d}} + (q - q^{-1})^{\vec{d}}(bc^{-1}e - aq^{-1}ek^{-1})^{\vec{d}}.$$

By (G2) the 0- and  $\vec{d}$ -homogeneous components of  $T_{\vec{d}}(A)^{\natural}$  are equal to

$$a^{\vec{d}}k^{-\vec{d}} + a^{-\vec{d}}k^{\vec{d}}, \quad (q - q^{-1})^{\vec{d}}(bc^{-1}e - aq^{-1}ek^{-1})^{\vec{d}}$$

respectively and the other homogeneous components of  $T_{\vec{d}}(A)^{\natural}$  are zero. To show the  $\vec{d}$ -homogeneous component of  $T_{\vec{d}}(A)$  as stated, one may simplify  $(bc^{-1}e - aq^{-1}ek^{-1})^{\vec{d}}$  by applying Proposition 2.9 with  $(R, S) = (bc^{-1}e, -aq^{-1}ek^{-1})$  followed by using Lemma 3.7 with  $n = \vec{d}$  and Lemma 3.8.

To see that  $T_{\vec{d}}(A)^{\natural}$  is equal to (19), make use of Lemma 3.6(ii) to express the homogeneous components of  $T_{\vec{d}}(A)^{\natural}$  in terms of  $x^{\vec{d}}, y^{\pm\vec{d}}, z^{\vec{d}}$ . The identities (20), (21) now follow by applying the automorphism  $\tilde{\rho}$  of  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U$  given in Lemma 2.7 to (19) and (20), respectively. Apply Lemma 3.6(i) to get the homogeneous components of  $T_{\vec{d}}(B)^{\natural}$  and  $T_{\vec{d}}(C)^{\natural}$ .  $\square$

We remark that Theorem 3.9 provides an alternative proof of Theorem 3.2 by the injectivity of  $\natural$ . Nevertheless this proof is lack of intuition.

#### 4. THE CENTER OF $\Delta$ AT ROOTS OF UNITY

**4.1. A basis of  $Z(\Delta)$ .** As seen in Theorem 3.2 the elements  $T_{\vec{d}}(A), T_{\vec{d}}(B), T_{\vec{d}}(C)$  are central in  $\Delta$ . In this subsection we shall show that

**Theorem 4.1.** *For each  $n \in \mathbb{N}$  the elements*

$$T_{\vec{d}}(A)^{i_0}T_{\vec{d}}(B)^{i_1}T_{\vec{d}}(C)^{i_2}\alpha^{j_0}\beta^{j_1}\gamma^{j_2}\Omega^{\ell}, \quad \begin{array}{l} i_0, i_1, i_2, j_0, j_1, j_2, \ell \in \mathbb{N}, \quad i_0i_1i_2 = 0, \\ \vec{d}(i_0 + i_1 + i_2) + j_0 + j_1 + j_2 + 3\ell \leq n \end{array}$$

*form an  $\mathbb{F}$ -basis of  $Z(\Delta) \cap \Delta_n$ . In particular the elements*

$$T_{\vec{d}}(A)^{i_0}T_{\vec{d}}(B)^{i_1}T_{\vec{d}}(C)^{i_2}\alpha^{j_0}\beta^{j_1}\gamma^{j_2}\Omega^{\ell} \quad \text{for all } i_0, i_1, i_2, j_0, j_1, j_2, \ell \in \mathbb{N} \quad \text{with } i_0i_1i_2 = 0$$

*form an  $\mathbb{F}$ -basis of  $Z(\Delta)$ .*

To prove Theorem 4.1 we resort to the  $\mathbb{F}$ -bases of  $\{\Delta_n\}_{n \in \mathbb{N}}$  as below. For each  $n \in \mathbb{N}$  define  $\mathbf{I}_n$  to be the set of all 10-tuples  $(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell)$  of nonnegative integers with

$$\begin{aligned} r_0 + r_1 + r_2 + \vec{d}(i_0 + i_1 + i_2) + j_0 + j_1 + j_2 + 3\ell &\leq n, \\ (r_0 + i_0)(r_1 + i_1)(r_2 + i_2) &= 0, \quad r_0, r_1, r_2 < \vec{d}. \end{aligned}$$

**Lemma 4.2.** *For each  $n \in \mathbb{N}$  the elements*

$$A^{r_0}B^{r_1}C^{r_2}T_{\vec{d}}(A)^{i_0}T_{\vec{d}}(B)^{i_1}T_{\vec{d}}(C)^{i_2}\alpha^{j_0}\beta^{j_1}\gamma^{j_2}\Omega^{\ell} \quad \text{for all } (r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell) \in \mathbf{I}_n$$

*form an  $\mathbb{F}$ -basis of  $\Delta_n$ .*

*Proof.* Regard  $\mathbb{F}[X]$  as a positional system with base  $T_d(X)$ . Namely, view each  $P(X) \in \mathbb{F}[X]$  as  $\sum_{i \in \mathbb{N}} P_i(X) T_d(X)^i$  with  $P_i(X) \in \mathbb{F}[X]$  of degree less than  $d$  and almost all zero. This lemma follows by applying the positional system to the  $\mathbb{F}$ -basis of  $\Delta_n$  given in Lemma 2.2.  $\square$

**Lemma 4.3.** *The following equations hold:*

$$\begin{aligned} CB &= q^2 BC & (\text{mod } \Delta_1), \\ CA &= q^{-2} AC & (\text{mod } \Delta_1), \\ BA &= q^2 AB & (\text{mod } \Delta_1). \end{aligned}$$

*Proof.* Immediate from (6)–(8).  $\square$

As usual, for any  $r, s$  in a ring the notation  $[r, s]$  is used to denote the commutator  $rs - sr$ .

**Lemma 4.4.** (i) *For all  $i_0, i_1, i_2 \in \mathbb{N}$*

$$\begin{aligned} [A^{i_0} B^{i_1} C^{i_2}, A] &= (q^{2(i_1-i_2)} - 1) A^{i_0+1} B^{i_1} C^{i_2} & (\text{mod } \Delta_{i_0+i_1+i_2}), \\ [A^{i_0} B^{i_1} C^{i_2}, B] &= (q^{2i_2} - q^{2i_0}) A^{i_0} B^{i_1+1} C^{i_2} & (\text{mod } \Delta_{i_0+i_1+i_2}), \\ [A^{i_0} B^{i_1} C^{i_2}, C] &= (1 - q^{2(i_1-i_0)}) A^{i_0} B^{i_1} C^{i_2+1} & (\text{mod } \Delta_{i_0+i_1+i_2}). \end{aligned}$$

(ii) *For all  $i_0, i_1, i_2 \in \mathbb{N}^*$*

$$A^{i_0} B^{i_1} C^{i_2} = (-1)^\ell q^{\ell(\ell-2i_1)} A^{i_0-\ell} B^{i_1-\ell} C^{i_2-\ell} \Omega^\ell \quad (\text{mod } \Delta_{i_0+i_1+i_2-1})$$

where  $\ell$  is any nonnegative integer less than or equal to each of  $i_0, i_1, i_2$ .

*Proof.* Property (F2) will be used frequently in the proof without reference. (i) follows by applying Lemma 4.3. Suppose  $i_0, i_1, i_2 \in \mathbb{N}^*$ . By Lemma 4.3 it yields that

$$A^{i_0} B^{i_1} C^{i_2} = q^{-2(i_1-1)} A^{i_0-1} B^{i_1-1} C^{i_2-1} ABC \quad (\text{mod } \Delta_{i_0+i_1+i_2-1}).$$

Observe from (9) that  $ABC = -q^{-1} \Omega \text{ mod } \Delta_2$ . Therefore

$$A^{i_0} B^{i_1} C^{i_2} = -q^{1-2i_1} A^{i_0-1} B^{i_1-1} C^{i_2-1} \Omega \quad (\text{mod } \Delta_{i_0+i_1+i_2-1}).$$

Now (ii) follows by a routine induction on  $i_0 + i_1 + i_2$ .  $\square$

It is ready to prove Theorem 4.1.

*Proof of Theorem 4.1.* The second statement is immediate from the first statement by (F1). Proceed by induction on  $n \in \mathbb{N}$  to prove the first statement. It is clear for  $n = 0$ . Now assume  $n \geq 1$ . Pick any  $R \in Z(\Delta) \cap \Delta_n$ . By Lemma 4.2 there exist unique  $c(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell) \in \mathbb{F}$  for all  $(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell) \in \mathbf{I}_n \setminus \mathbf{I}_{n-1}$  such that  $R \text{ mod } \Delta_{n-1}$  is equal to

$$S = \sum_{(r_0, r_1, \dots, \ell)} c(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell) A^{r_0} B^{r_1} C^{r_2} T_d(A)^{i_0} T_d(B)^{i_1} T_d(C)^{i_2} \alpha^{j_0} \beta^{j_1} \gamma^{j_2} \Omega^\ell$$

where the sum is over all  $(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell) \in \mathbf{I}_n \setminus \mathbf{I}_{n-1}$ . We shall see that if at least one of  $r_0, r_1, r_2$  is positive then  $c(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell) = 0$ . If this is true then  $S \in Z(\Delta)$  by Theorem 3.2 and this theorem follows by applying induction hypothesis to  $R - S \in Z(\Delta) \cap \Delta_{n-1}$ . It remains to prove the claim.

Pick any  $c(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell)$  with at least one of  $r_0, r_1, r_2$  in  $\mathbb{N}^*$ . Without loss assume that  $r_0 \in \mathbb{N}^*$ . By the definition of  $\mathbf{I}_n$ , at least one of  $(r_1, i_1), (r_2, i_2)$  is equal to  $(0, 0)$ , say  $(r_1, i_1)$ . In this case, consider the commutator  $[S, C]$ . Since  $R$  is central in  $\Delta$  the commutator  $[R, C] = 0$ . It follows from (F2) that

$$[S, C] \in \Delta_n.$$

On the other hand, applying Lemma 4.4 the coefficient of  $A^{r_0+\vec{d}i_0}C^{r_2+\vec{d}i_2+1}\alpha^{j_0}\beta^{j_1}\gamma^{j_2}\Omega^\ell$  in  $[S, C]$  with respect to the  $\mathbb{F}$ -basis of  $\Delta_{n+1}$  given in Lemma 2.2 is equal to

$$(1 - q^{-2r_0}) \cdot c(r_0, r_1, r_2, i_0, i_1, i_2, j_0, j_1, j_2, \ell).$$

Since  $[S, C] \in \Delta_n$  the above is equal to 0. Recall that the order of the root of unity  $q^2$  is equal to  $\vec{d}$ . Therefore  $q^{2r_0} \neq 1$  and this forces that the claim is true. The result follows.  $\square$

**4.2. A relation in  $Z(\Delta)$ .** The goal of this subsection is to prove Theorems 4.5 and 4.6 stated below. For each  $n \in \mathbb{N}$  let

$$\begin{aligned} \phi_n(X_0, X_1, X_2; X) &= T_n(X)T_n(X_0) + T_n(X_1)T_n(X_2), \\ \psi_n(X_0, X_1, X_2; X) &= T_{2n}(X) + T_n(X_0)^2 + T_n(X_1)^2 + T_n(X_2)^2 + T_n(X)T_n(X_0)T_n(X_1)T_n(X_2). \end{aligned}$$

To each  $i \in \mathbb{Z}/3\mathbb{Z}$  we associate an  $\mathbb{F}[X]$ -algebra automorphism  $\underline{i}$  of  $\mathbb{F}[X_0, X_1, X_2, X]$  with

$$X_j^{\underline{i}} = X_{i+j} \quad \text{for all } j \in \mathbb{Z}/3\mathbb{Z}.$$

**Theorem 4.5.** *For each  $n \in \mathbb{N}$  there exist unique polynomials  $\Phi_n(X_0, X_1, X_2; X)$ ,  $\Psi_n(X_0, X_1, X_2; X)$  over  $\mathbb{Z}$  satisfying*

$$\Phi_n(\phi_m^0, \phi_m^1, \phi_m^2; \psi_m) = \phi_{mn}, \quad \Psi_n(\phi_m^0, \phi_m^1, \phi_m^2; \psi_m) = \psi_{mn} \quad \text{for all } m \in \mathbb{N}.$$

**Theorem 4.6.** *The relation*

$$\begin{aligned} &q^{\vec{d}}\Phi_{\vec{d}}^0(\alpha, \beta, \gamma; \Omega)T_{\vec{d}}(A) + q^{\vec{d}}\Phi_{\vec{d}}^1(\alpha, \beta, \gamma; \Omega)T_{\vec{d}}(B) + q^{\vec{d}}\Phi_{\vec{d}}^2(\alpha, \beta, \gamma; \Omega)T_{\vec{d}}(C) \\ &= q^{\vec{d}}T_{\vec{d}}(A)T_{\vec{d}}(B)T_{\vec{d}}(C) + T_{\vec{d}}(A)^2 + T_{\vec{d}}(B)^2 + T_{\vec{d}}(C)^2 + \Psi_{\vec{d}}(\alpha, \beta, \gamma; \Omega) - 2 \end{aligned}$$

*holds in  $Z(\Delta)$ .*

Before launching into the proof of Theorems 4.5 and 4.6 we make some preparations.

**Lemma 4.7.** *For any  $r_0, r_1, r_2, s \in \mathbb{N}$ , if there are  $i_0, i_1, i_2, j \in \mathbb{N}$  such that the degree of*

$$(\phi_1^0)^{i_0}(\phi_1^1)^{i_1}(\phi_1^2)^{i_2}(\psi_1)^j$$

*as a polynomial of  $X$  is equal to  $r_0 + r_1 + r_2 + 2s$  with leading coefficient  $X_0^{r_0}X_1^{r_1}X_2^{r_2}$ , then  $(i_0, i_1, i_2, j) = (r_0, r_1, r_2, s)$ .*

*Proof.* Expand  $(\phi_1^0)^{i_0}(\phi_1^1)^{i_1}(\phi_1^2)^{i_2}(\psi_1)^j$  directly.  $\square$

**Lemma 4.8.** *For each  $n \in \mathbb{N}$  there exists a unique  $\mathbb{F}$ -algebra isomorphism  $\mathbb{F}[X_0, X_1, X_2, X] \rightarrow \mathbb{F}[T_n(X_0), T_n(X_1), T_n(X_2), T_n(X)]$  that sends*

$$X_0 \mapsto T_n(X_0), \quad X_1 \mapsto T_n(X_1), \quad X_2 \mapsto T_n(X_2), \quad X \mapsto T_n(X).$$

*Moreover this isomorphism maps  $\phi_m$  to  $\phi_{mn}$  and  $\psi_m$  to  $\psi_{mn}$  for all  $m \in \mathbb{N}$ .*

*Proof.* Lemma 2.10(i) implies the algebraic independence of  $T_n(X_0), T_n(X_1), T_n(X_2), T_n(X)$  over  $\mathbb{F}$ . Therefore the desired isomorphism exists. Applying Lemma 2.10(ii) it yields that

$$\begin{aligned} \phi_m(T_n(X_0), T_n(X_1), T_n(X_2); T_n(X)) &= \phi_{mn}(X_0, X_1, X_2; X), \\ \psi_m(T_n(X_0), T_n(X_1), T_n(X_2); T_n(X)) &= \psi_{mn}(X_0, X_1, X_2; X). \end{aligned}$$

This lemma follows.  $\square$

**Lemma 4.9.** *For each  $n \in \mathbb{N}^*$  the polynomials  $\phi_n^0, \phi_n^1, \phi_n^2, \psi_n$  are algebraically independent over  $\mathbb{F}$ .*

*Proof.* Consider the equation

$$(22) \quad \sum_{i_0, i_1, i_2, j \in \mathbb{N}} c_{i_0, i_1, i_2, j} (\phi_1^0)^{i_0} (\phi_1^1)^{i_1} (\phi_1^2)^{i_2} (\psi_1)^j = 0$$

with  $c_{i_0, i_1, i_2, j} \in \mathbb{F}$  and almost all zero. Suppose there exist  $r_0, r_1, r_2, s \in \mathbb{N}$  with  $c_{r_0, r_1, r_2, s} \neq 0$  and other coefficients  $c_{i_0, i_1, i_2, j} \neq 0$  only if  $r_0 + r_1 + r_2 + 2s \geq i_0 + i_1 + i_2 + 2j$ . By Lemma 4.7 the coefficient of  $X_0^{r_0} X_1^{r_1} X_2^{r_2} X^{r_0 + r_1 + r_2 + 2s}$  in the left-hand side of (22) is equal to  $c_{r_0, r_1, r_2, s}$ , a contradiction to (22). Therefore  $c_{i_0, i_1, i_2, j} = 0$  for all  $i_0, i_1, i_2, j \in \mathbb{N}$ . This shows that  $\phi_1^0, \phi_1^1, \phi_1^2, \psi_1$  are algebraically independent over  $\mathbb{F}$ . For any  $n \in \mathbb{N}^*$  the algebraic independence of  $\phi_n^0, \phi_n^1, \phi_n^2, \psi_n$  now follows from Lemma 4.8.  $\square$

**Proposition 4.10.** *There exist two unique polynomials  $\Phi(X_0, X_1, X_2; X), \Psi(X_0, X_1, X_2; X)$  over  $\mathbb{F}$  satisfying*

$$\Phi(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \phi_d, \quad \Psi(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \psi_d.$$

Moreover

$$\begin{aligned} & q^d \Phi^0(\alpha, \beta, \gamma; \Omega) T_d(A) + q^d \Phi^1(\alpha, \beta, \gamma; \Omega) T_d(B) + q^d \Phi^2(\alpha, \beta, \gamma; \Omega) T_d(C) \\ &= q^d T_d(A) T_d(B) T_d(C) + T_d(A)^2 + T_d(B)^2 + T_d(C)^2 + \Psi(\alpha, \beta, \gamma; \Omega) - 2. \end{aligned}$$

*Proof.* By Lemma 4.9 the polynomials  $\Phi(X_0, X_1, X_2; X)$  and  $\Psi(X_0, X_1, X_2; X)$  are unique if they exist. To see the existence consider the central element

$$T_d(A) T_d(B) T_d(C).$$

By Lemma 4.4(ii) with  $i_0 = i_1 = i_2 = d$ , it follows that  $T_d(A) T_d(B) T_d(C) - (-1)^d q^{-d^2} \Omega^d \in \Delta_{3d-1}$ . By Theorem 4.1 there are unique  $\Sigma_A, \Sigma_B, \Sigma_C, \Xi_A, \Xi_B, \Xi_C, \Phi_A, \Phi_B, \Phi_C, \Psi \in \mathbb{F}[\alpha, \beta, \gamma, \Omega]$  such that  $T_d(A) T_d(B) T_d(C)$  is equal to

$$(23) \quad \begin{aligned} & \Sigma_A T_d(B) T_d(C) + \Sigma_B T_d(C) T_d(A) + \Sigma_C T_d(A) T_d(B) + \Xi_A T_d(A)^2 + \Xi_B T_d(B)^2 + \Xi_C T_d(C)^2 \\ &+ \Phi_A T_d(A) + \Phi_B T_d(B) + \Phi_C T_d(C) + \Psi. \end{aligned}$$

For  $d = 1$  it follows from (9) that

$$\begin{aligned} \Sigma_A = \Sigma_B = \Sigma_C &= 0, & \Xi_A = \Xi_B = \Xi_C &= -q, \\ \Phi_A = \alpha, & \Phi_B = \beta, & \Phi_C = \gamma, & \Psi = q(2 - \Omega). \end{aligned}$$

Thus  $\Phi(X_0, X_1, X_2; X) = X_0$  and  $\Psi(X_0, X_1, X_2; X) = X$  when  $d = 1$ .

Assume that  $d > 1$ . By Lemma 2.12(ii),  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} Z(U)$  has the  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}[\Lambda]$ -basis

$$(24) \quad k^{\pm di} f^j, \quad k^{\pm di}, \quad k^{\pm di} e^j \quad \text{for all } i \in \mathbb{N} \text{ and } j \in \mathbb{N}^*.$$

By Lemma 2.6 the images of  $\Sigma_A, \Sigma_B, \Sigma_C, \Xi_A, \Xi_B, \Xi_C, \Phi_A, \Phi_B, \Phi_C, \Psi$  under  $\natural$  are contained in  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}[\Lambda]$ . To find the values of them, express

$$\begin{aligned} & T_d(A)^{\natural}, \quad T_d(B)^{\natural}, \quad T_d(C)^{\natural}, \quad T_d(A)^{2\natural}, \quad T_d(B)^{2\natural}, \quad T_d(C)^{2\natural}, \\ & T_d(A)^{\natural} T_d(B)^{\natural}, \quad T_d(B)^{\natural} T_d(C)^{\natural}, \quad T_d(C)^{\natural} T_d(A)^{\natural}, \quad T_d(A)^{\natural} T_d(B)^{\natural} T_d(C)^{\natural} \end{aligned}$$

as linear combinations of (24). The results of calculation are included in the appendix of this paper. The coefficient of  $e^{2d}$  in  $T_d(A)^{\natural} T_d(B)^{\natural} T_d(C)^{\natural}$  is equal to

$$-(q - q^{-1})^{2d} q^d b^{2d} c^{-2d}.$$

Denote by  $(23)^{\natural}$  the equation obtained by applying  $\natural$  to (23). Therefore the coefficient of  $e^{2\bar{d}}$  in  $(23)^{\natural}$  is equal to

$$(q - q^{-1})^{2\bar{d}} b^{2\bar{d}} c^{-2\bar{d}} \Xi_A^{\natural}.$$

This leads to  $\Xi_A = -q^{\bar{d}}$ . The comparison of the coefficients of  $k^{-\bar{d}} e^{2\bar{d}}$  yields that  $\Sigma_B = 0$ . Since  $T_{\bar{d}}(A)T_{\bar{d}}(B)T_{\bar{d}}(C)$  is fixed by  $\rho$  the automorphism  $\rho$  sends

$$\begin{aligned} (\Sigma_A, \Sigma_B, \Sigma_C) &\mapsto (\Sigma_B, \Sigma_C, \Sigma_A), \\ (\Xi_A, \Xi_B, \Xi_C) &\mapsto (\Xi_B, \Xi_C, \Xi_A), \\ (\Phi_A, \Phi_B, \Phi_C) &\mapsto (\Phi_B, \Phi_C, \Phi_A). \end{aligned}$$

As a result

$$\Sigma_A = \Sigma_B = \Sigma_C = 0, \quad \Xi_A = \Xi_B = \Xi_C = -q^{\bar{d}}.$$

Now, the comparison of the coefficients of  $e^{\bar{d}}$  in  $T_{\bar{d}}(A)^{\natural}T_{\bar{d}}(B)^{\natural}T_{\bar{d}}(C)^{\natural}$  and  $(23)^{\natural}$  determines that

$$\Phi_A^{\natural} = T_{\bar{d}}(\Lambda)(a^{\bar{d}} + a^{-\bar{d}}) + (b^{\bar{d}} + b^{-\bar{d}})(c^{\bar{d}} + c^{-\bar{d}}).$$

By Lemma 2.7 the automorphism  $\tilde{\rho}$  of  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} U'$  cyclically permutes  $\Phi_A^{\natural}, \Phi_B^{\natural}, \Phi_C^{\natural}$ . Combined with Lemma 2.8 it follows that

$$\begin{aligned} \Phi_B^{\natural} &= T_{\bar{d}}(\Lambda)(b^{\bar{d}} + b^{-\bar{d}}) + (c^{\bar{d}} + c^{-\bar{d}})(a^{\bar{d}} + a^{-\bar{d}}), \\ \Phi_C^{\natural} &= T_{\bar{d}}(\Lambda)(c^{\bar{d}} + c^{-\bar{d}}) + (a^{\bar{d}} + a^{-\bar{d}})(b^{\bar{d}} + b^{-\bar{d}}). \end{aligned}$$

Comparing the coefficients of 1 it follows that  $\Psi^{\natural}$  is equal to  $-q^{\bar{d}}$  times

$$T_{2\bar{d}}(\Lambda) + (a^{\bar{d}} + a^{-\bar{d}})^2 + (b^{\bar{d}} + b^{-\bar{d}})^2 + (c^{\bar{d}} + c^{-\bar{d}})^2 + T_{\bar{d}}(\Lambda)(a^{\bar{d}} + a^{-\bar{d}})(b^{\bar{d}} + b^{-\bar{d}})(c^{\bar{d}} + c^{-\bar{d}}) - 2.$$

Normalize  $\Psi$  by multiplying the factor  $-q^{\bar{d}}$  followed by adding 2. Abuse the notation  $\Psi$  to denote the normalized one. By Lemma 2.10(i) we may write

$$\begin{aligned} \Phi_A^{\natural} &= \phi_{\bar{d}}(a + a^{-1}, b + b^{-1}, c + c^{-1}; \Lambda), \\ \Psi^{\natural} &= \psi_{\bar{d}}(a + a^{-1}, b + b^{-1}, c + c^{-1}; \Lambda). \end{aligned}$$

Since  $\Phi_A, \Psi \in \mathbb{F}[\alpha, \beta, \gamma; \Omega]$  there exist two polynomials  $\Phi(X_0, X_1, X_2; X), \Psi(X_0, X_1, X_2; X)$  over  $\mathbb{F}$  with  $\Phi(\alpha^{\natural}, \beta^{\natural}, \gamma^{\natural}; \Omega^{\natural}) = \Phi_A^{\natural}$  and  $\Psi(\alpha^{\natural}, \beta^{\natural}, \gamma^{\natural}; \Omega^{\natural}) = \Psi^{\natural}$ . Observe from Lemma 2.5 that

$$\begin{aligned} \alpha^{\natural} &= \phi_1^0(a + a^{-1}, b + b^{-1}, c + c^{-1}; \Lambda), \\ \beta^{\natural} &= \phi_1^1(a + a^{-1}, b + b^{-1}, c + c^{-1}; \Lambda), \\ \gamma^{\natural} &= \phi_1^2(a + a^{-1}, b + b^{-1}, c + c^{-1}; \Lambda), \\ \Omega^{\natural} &= \psi_1(a + a^{-1}, b + b^{-1}, c + c^{-1}; \Lambda). \end{aligned}$$

By Lemma 2.12(i),  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}[\Lambda]$  has the  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ -basis  $\{\Lambda^n\}_{n \in \mathbb{N}}$ . In other words  $\Lambda$  is algebraically independent over  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ . Together with the algebraic independence of  $a + a^{-1}, b + b^{-1}, c + c^{-1}$  over  $\mathbb{F}$ , the elements

$$a + a^{-1}, \quad b + b^{-1}, \quad c + c^{-1}, \quad \Lambda$$

are algebraically independent over  $\mathbb{F}$ . Concluding from the above results the two polynomials  $\Phi(X_0, X_1, X_2; X), \Psi(X_0, X_1, X_2; X)$  satisfy the desired properties.  $\square$

It is in the position to prove Theorems 4.5 and 4.6.

*Proof of Theorem 4.5.* Set  $\mathbb{F} = \mathbb{C}$  and consider the polynomials  $\Phi(X_0, X_1, X_2; X), \Psi(X_0, X_1, X_2; X)$  from Proposition 4.10.

Suppose some coefficients of  $\Phi(X_0, X_1, X_2; X)$  are not integers. Choose a monomial  $X_0^{r_0} X_1^{r_1} X_2^{r_2} X^s$  in  $\Phi(X_0, X_1, X_2; X)$  with non-integral coefficient that satisfies the property: For any  $i_0, i_1, i_2, j \in \mathbb{N}$ ,



if the coefficient of  $X_0^{i_0} X_1^{i_1} X_2^{i_2} X^j$  in  $\Phi(X_0, X_1, X_2; X)$  is not an integer, then  $r_0 + r_1 + r_2 + 2s \geq i_0 + i_1 + i_2 + 2j$ . Substitute  $(X_0, X_1, X_2, X) = (\phi_1^0, \phi_1^1, \phi_1^2, \psi_1)$  into  $\Phi(X_0, X_1, X_2; X)$ . By Lemma 4.7 the coefficient of  $X_0^{r_0} X_1^{r_1} X_2^{r_2} X^{r_0+r_1+r_2+2s}$  in  $\Phi(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1)$  is not an integer. This contradicts that  $\Phi(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \phi_d$  is a polynomial over  $\mathbb{Z}$ . Therefore the coefficients of  $\Phi(X_0, X_1, X_2; X)$  are integers. Let  $m \in \mathbb{N}$  be given. Substitute  $X_i = T_m(X_i)$  for all  $i \in \mathbb{Z}/3\mathbb{Z}$  and  $X = T_m(X)$  into  $\Phi(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \phi_d$ . By Lemma 4.8 it follows that  $\Phi(\phi_m^0, \phi_m^1, \phi_m^2; \psi_m) = \phi_{md}$ . We have shown that  $\Phi(X_0, X_1, X_2; X) = \Phi_d(X_0, X_1, X_2; X)$ . By a similar argument  $\Psi(X_0, X_1, X_2; X) = \Psi_d(X_0, X_1, X_2; X)$ .

Given any positive integer  $d$  there always exists an element  $q \in \mathbb{F}$  to be a primitive  $d^{\text{th}}$  root of unity since  $\mathbb{F}$  is now assumed to be  $\mathbb{C}$ . Therefore  $\vec{d}$  could be any positive integer. The result follows.  $\square$

*Proof of Theorem 4.6.* By Theorem 4.5 the polynomials  $\Phi_d(X_0, X_1, X_2; X)$ ,  $\Psi_d(X_0, X_1, X_2; X)$  satisfy  $\Phi_d(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \phi_d$ ,  $\Psi_d(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \psi_d$ . Thus, for any ground field  $\mathbb{F}$  the polynomials  $\Phi(X_0, X_1, X_2; X)$ ,  $\Psi(X_0, X_1, X_2; X)$  from Proposition 4.10 are equal to  $\Phi_d(X_0, X_1, X_2; X)$ ,  $\Psi_d(X_0, X_1, X_2; X)$  respectively.  $\square$

Some properties about  $\{\Phi_n\}_{n \in \mathbb{N}}$  and  $\{\Psi_n\}_{n \in \mathbb{N}}$  are worthy of mention.

**Lemma 4.11.** *There exists a unique  $\mathbb{F}$ -algebra isomorphism  $\mathbb{F}[X_0, X_1, X_2, X] \rightarrow \mathbb{F}[\phi_1^0, \phi_1^1, \phi_1^2, \psi_1]$  that sends*

$$\Phi_n^i \mapsto \phi_n^i, \quad \Psi_n^i \mapsto \psi_n \quad \text{for all } n \in \mathbb{N} \text{ and all } i \in \mathbb{Z}/3\mathbb{Z}.$$

*Proof.* By construction  $\psi_n = \psi_n^i$  for all  $n \in \mathbb{N}$  and all  $i \in \mathbb{Z}/3\mathbb{Z}$ . By Lemma 4.9 there is a unique  $\mathbb{F}$ -algebra isomorphism  $\mathbb{F}[X_0, X_1, X_2, X] \rightarrow \mathbb{F}[\phi_1^0, \phi_1^1, \phi_1^2, \psi_1]$  that maps  $X_i$  to  $\phi_1^i$  for all  $i \in \mathbb{Z}/3\mathbb{Z}$  and  $X$  to  $\psi_1$ . This isomorphism satisfies the desired property due to Theorem 4.5.  $\square$

As consequences of Lemma 4.11 we have

**Proposition 4.12.** (i)  $\Psi_n^i = \Psi_n$  for all  $n \in \mathbb{N}$  and  $i \in \mathbb{Z}/3\mathbb{Z}$ .

(ii)  $\Phi_n(\Phi_m^0, \Phi_m^1, \Phi_m^2; \Psi_m) = \Phi_{mn}$  and  $\Psi_n(\Phi_m^0, \Phi_m^1, \Phi_m^2; \Psi_m) = \Psi_{mn}$  for all  $m, n \in \mathbb{N}$ .

(iii)  $\Phi_n^0, \Phi_n^1, \Phi_n^2, \Psi_n$  are algebraically independent over  $\mathbb{F}$  for each  $n \in \mathbb{N}^*$ .

*Proof.* (i) is trivial. (ii) follows from Theorem 4.5. (iii) follows from Lemma 4.9.  $\square$

**4.3. A presentation for  $Z(\Delta)$ .** As a consequence of Theorem 4.1 the elements  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\Omega$  form a set of generators of  $Z(\Delta)$ . This subsection is devoted to proving that

**Theorem 4.13.** *The center of  $\Delta$  is the commutative  $\mathbb{F}$ -algebra generated by  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\Omega$  subject to the relation*

$$\begin{aligned} & q^{\vec{d}} \Phi_d^0(\alpha, \beta, \gamma; \Omega) T_d(A) + q^{\vec{d}} \Phi_d^1(\alpha, \beta, \gamma; \Omega) T_d(B) + q^{\vec{d}} \Phi_d^2(\alpha, \beta, \gamma; \Omega) T_d(C) \\ &= q^{\vec{d}} T_d(A) T_d(B) T_d(C) + T_d(A)^2 + T_d(B)^2 + T_d(C)^2 + \Psi_d(\alpha, \beta, \gamma; \Omega) - 2. \end{aligned}$$

*Proof.* Let  $K$  denote the  $\mathbb{F}$ -subalgebra of  $Z(\Delta)$  generated by  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ . Consider the  $K$ -algebra homomorphism  $K[X] \rightarrow Z(\Delta)$  that maps  $X \mapsto \Omega$ . Since  $Z(\Delta) = K[\Omega]$  by Theorem 4.1, the homomorphism is surjective. By Theorem 4.6 the kernel of this homomorphism contains the ideal of  $K[X]$  generated by

$$\begin{aligned} P(X) &= \Psi_d(\alpha, \beta, \gamma; X) - q^{\vec{d}} \Phi_d^0(\alpha, \beta, \gamma; X) T_d(A) - q^{\vec{d}} \Phi_d^1(\alpha, \beta, \gamma; X) T_d(B) - q^{\vec{d}} \Phi_d^2(\alpha, \beta, \gamma; X) T_d(C) \\ &+ q^{\vec{d}} T_d(A) T_d(B) T_d(C) + T_d(A)^2 + T_d(B)^2 + T_d(C)^2 - 2. \end{aligned}$$

It suffices to show the converse inclusion. Furthermore, we shall show that  $P(X)$  is equal to the minimal polynomial  $R(X)$  of  $\Omega$  over the fraction field of  $K$ . We begin with a lemma which implies that  $P(X)$  is a monic polynomial of degree  $\vec{d}$ .

**Lemma 4.14.** *As polynomials of  $X$ ,*

- (i)  $\Phi_n(X_0, X_1, X_2; X)$  *is of degree less than or equal to  $\lfloor n/2 \rfloor$  for each  $n \in \mathbb{N}$ ;*
- (ii)  $\Psi_n(X_0, X_1, X_2; X)$  *is monic of degree  $n$  for each  $n \in \mathbb{N}^*$ .*

*Proof.* By Theorem 4.5,  $\Phi_n(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \phi_n$  is of degree  $n$  as a polynomial of  $X$ . Therefore, by Lemma 4.7 each monomial  $X_0^{r_0} X_1^{r_1} X_2^{r_2} X^s$  in  $\Phi_n(X_0, X_1, X_2; X)$  with nonzero coefficient satisfies  $r_0 + r_1 + r_2 + 2s \leq n$  and in particular  $2s \leq n$ . This shows (i).

By Theorem 4.5,  $\Psi_n(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1) = \psi_n$  is monic of degree  $2n$  as a polynomial of  $X$ . By Lemma 4.7 each monomial  $X_0^{r_0} X_1^{r_1} X_2^{r_2} X^s$  in  $\Psi_n(X_0, X_1, X_2; X)$  with nonzero coefficient satisfies  $r_0 + r_1 + r_2 + 2s \leq 2n$  and the equality only holds for the monomial  $X^n$  with coefficient 1. This shows (ii).  $\square$

Recall that  $\rho$  sends

$$(A, B, C, \alpha, \beta, \gamma) \mapsto (B, C, A, \beta, \gamma, \alpha).$$

Hence  $K$  is  $\rho$ -invariant. Here we abuse the notation  $\rho$  to denote the  $\mathbb{F}$ -algebra automorphism  $K[X] \rightarrow K[X]$  induced by  $\rho$  that leaves  $X$  fixed. By Proposition 4.12(i) a direct calculation yields that  $P(X)$  is  $\rho$ -invariant.

**Lemma 4.15.** *Let  $Q(X)$  denote the quotient of  $P(X)$  divided by  $R(X)$ . Then the coefficients of  $Q(X)$ ,  $R(X)$  lie in  $K$  and each of  $Q(X)$ ,  $R(X)$  is  $\rho$ -invariant.*

*Proof.* By Lemma 4.14 the polynomial  $P(X)$  is monic over  $K$  and hence the roots of  $P(X)$  are integral over  $K$ . Since  $R(X)$  divides  $P(X)$  the roots of  $R(X)$  are integral over  $K$  and so are the coefficients of  $R(X)$ . By Lemma 2.1,  $K$  is a six-variable polynomial ring over  $\mathbb{F}$  generated by  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ . In particular  $K$  is integrally closed. Therefore  $R(X) \in K[X]$  and so is  $Q(X)$ .

Since  $\Omega^\rho = \Omega$  by Lemma 2.3, the element  $\Omega$  is a root of  $R(X)^\rho$ . Therefore  $R(X)$  is  $\rho$ -invariant by the uniqueness of  $R(X)$ . Applying  $\rho$  to  $P(X) = Q(X)R(X)$  it follows that  $Q(X)$  is  $\rho$ -invariant by the uniqueness of  $Q(X)$ .  $\square$

Let  $L$  denote the  $\mathbb{F}$ -subalgebra of  $K$  generated by  $\alpha$ ,  $\beta$ ,  $\gamma$ . For each  $g \in K$  let  $\deg g$  denote the degree of  $g$  as a polynomial of  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$  over  $L$ . Write a polynomial  $G(X) \in K[X]$  as  $\sum_{i \in \mathbb{N}} g_i X^i$  with all  $g_i \in K$  and almost all zero. The coefficient  $g_n$  is said to be *chief* if  $n$  is the maximal nonnegative integer satisfying

$$\deg g_i \leq \deg g_n \quad \text{for all } i \in \mathbb{N}.$$

**Lemma 4.16.** *Given  $F(X), G(X) \in K[X]$ , if the coefficients of  $X^m, X^n$  in  $F(X), G(X)$  are chief with degrees  $r, s$  respectively, then the coefficient of  $X^{m+n}$  in  $F(X)G(X)$  is chief with degree  $r + s$ .*

*Proof.* Write  $F(x) = \sum_{i \in \mathbb{N}} f_i X^i$  and  $G(X) = \sum_{i \in \mathbb{N}} g_i X^i$  with  $f_i, g_i$  in  $K$  and almost all zero. Then

$$\deg f_i + \deg g_j \leq r + s \quad \text{for all } i, j \in \mathbb{N}.$$

The equality holds if  $i = m$  and  $j = n$  and only if  $i \leq m$  and  $j \leq n$ . The result follows.  $\square$

From the definition of  $P(X)$  we see that the constant term of  $P(X)$  is chief with degree 3. By Lemma 4.16 with  $(F(X), G(X)) = (Q(X), R(X))$  the constant term  $q_0$  of  $Q(X)$  is chief and so is the constant term  $r_0$  of  $R(X)$ . Observe that the constant term of  $P(X)$  is not equal to a product of two  $\rho$ -invariant elements in  $K$  that have degrees 1 and 2. Therefore  $\deg r_0$  is equal to 0 or 3 by Lemma 4.15. Since  $R(X)$  is of positive degree this forces that  $\deg r_0 = 3$  and hence  $\deg q_0 = 0$ . This shows that  $Q(X)$  is a constant. Since  $P(X)$  and  $R(X)$  are monic polynomials it follows that  $P(X) = R(X)$ .  $\square$

Replace the roles of  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$  in Theorem 4.13 by  $q^{\vec{d}}T_d(A)$ ,  $q^{\vec{d}}T_d(B)$ ,  $q^{\vec{d}}T_d(C)$  respectively. Combining with Lemma 4.8 and Lemma 4.11 it follows that

**Theorem 4.17.**  $Z(\Delta)$  is isomorphic to the polynomial ring  $\mathbb{F}[X_0, X_1, X_2, X, Y_0, Y_1, Y_2]$  modulo the ideal generated by

$$(25) \quad \prod_{i \in \mathbb{Z}/3\mathbb{Z}} Y_i + \sum_{i \in \mathbb{Z}/3\mathbb{Z}} Y_i^2 - \sum_{i \in \mathbb{Z}/3\mathbb{Z}} \phi_1^i(X_0, X_1, X_2; X) Y_i + \psi_1(X_0, X_1, X_2; X) - 2.$$

In the proof of Theorem 4.13 we also see that

**Theorem 4.18.** *The elements*

$$T_d(A)^{i_0} T_d(B)^{i_1} T_d(C)^{i_2} \alpha^{j_0} \beta^{j_1} \gamma^{j_2} \Omega^\ell \quad \text{for all } i_0, i_1, i_2, j_0, j_1, j_2, \ell \in \mathbb{N} \text{ with } \ell < \vec{d}$$

form an  $\mathbb{F}$ -basis of  $Z(\Delta)$ .

**Theorem 4.19.**  $Z(\Delta)$  is integral over the  $\mathbb{F}$ -subalgebra of  $Z(\Delta)$  generated by  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ .

We end this subsection with a feedback to  $Z(U'_q(\mathfrak{so}_3))$ . By the construction of  $\Delta$  there is an  $\mathbb{F}$ -algebra homomorphism  $\iota : \Delta \rightarrow U'_q(\mathfrak{so}_3)$  that sends

$$(A, B, C, \alpha, \beta, \gamma, \Omega) \mapsto (-(q^2 - q^{-2})K_0, -(q^2 - q^{-2})K_1, -(q^2 - q^{-2})K_2, 0, 0, 0, \Pi).$$

Recall from Introduction that  $\Gamma_i$  denotes  $T_d(-(q^2 - q^{-2})K_i)$  for each  $i \in \mathbb{Z}/3\mathbb{Z}$ . Hence  $\iota$  maps  $T_d(A)$ ,  $T_d(B)$ ,  $T_d(C)$  to  $\Gamma_0$ ,  $\Gamma_1$ ,  $\Gamma_2$  respectively.

From the definitions of  $\phi_n(X_0, X_1, X_2; X)$ ,  $\psi_n(X_0, X_1, X_2; X)$  we see that

$$\begin{aligned} \phi_n(0, 0, 0; X) &= 2 \left( \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left( (-1)^{\lfloor n/2 \rfloor} T_n(X) + 2 \right), \\ \psi_n(0, 0, 0; X) &= T_{2n}(X) + 4 \left( \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left( (-1)^{\lfloor n/2 \rfloor} 2 T_n(X) + 3 \right) \end{aligned}$$

for all  $n \in \mathbb{N}$ . Substitute  $(X_0, X_1, X_2) = (0, 0, 0)$  into  $\Phi_n(\phi_1^0, \phi_1^1, \phi_1^2; \psi_1)$ . By Theorem 4.5 it follows that

$$\Phi_n(0, 0, 0; T_2(X)) = 2 \left( \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left( (-1)^{\lfloor n/2 \rfloor} T_n(X) + 2 \right) \quad \text{for all } n \in \mathbb{N}.$$

Using Lemma 2.10(ii) it follows that

$$\Phi_n(0, 0, 0; X) = 2 \left( \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left( (-1)^{\lfloor n/2 \rfloor} T_{\lfloor n/2 \rfloor}(X) + 2 \right) \quad \text{for all } n \in \mathbb{N}.$$

A similar way shows that

$$\Psi_n(0, 0, 0; X) = T_n(X) + 4 \left( \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil + 1 \right) \left( (-1)^{\lfloor n/2 \rfloor} 2 T_{\lfloor n/2 \rfloor}(X) + 3 \right) \quad \text{for all } n \in \mathbb{N}.$$

Applying the above results, the image of the relation in Theorem 4.6 under  $\iota$  implies the relation (5). By a similar proof to that of Theorem 4.13, the presentation for  $Z(U'_q(\mathfrak{so}_3))$  conjectured by Iorgov [16, Conjecture 1] can be shown to be true.

## 5. THE CENTER OF DAHA OF TYPE $(C_1^\vee, C_1)$ AT ROOTS OF UNITY

Fix four nonzero parameters  $k_0$ ,  $k_1$ ,  $k_0^\vee$ ,  $k_1^\vee$  taken from  $\mathbb{F}$ . The DAHA of type  $(C_1^\vee, C_1)$  is an algebra  $\mathfrak{H} = \mathfrak{H}_q(k_0, k_1, k_0^\vee, k_1^\vee)$  generated by  $t_0$ ,  $t_1$ ,  $t_0^\vee$ ,  $t_1^\vee$  subject to the relations

$$(26) \quad \begin{aligned} (t_0 - k_0)(t_0 - k_0^{-1}) &= 0, & (t_1 - k_1)(t_1 - k_1^{-1}) &= 0, \\ (t_0^\vee - k_0^\vee)(t_0^\vee - k_0^{\vee-1}) &= 0, & (t_1^\vee - k_1^\vee)(t_1^\vee - k_1^{\vee-1}) &= 0, \end{aligned}$$

$$(27) \quad t_0^\vee t_0 t_1^\vee t_1 = q^{-1}.$$

The goal of §5 is to give a presentation for  $Z(\mathfrak{H})$ .

5.1. **An  $\mathbb{N}$ -filtration structure of  $\mathfrak{H}$ .** Let  $c_0, c_1, c_0^\vee, c_1^\vee$  denote the scalars

$$k_0 + k_0^{-1}, \quad k_1 + k_1^{-1}, \quad k_0^\vee + k_0^{\vee-1}, \quad k_1^\vee + k_1^{\vee-1}$$

respectively. By (26) the generators  $t_i$  and  $t_i^\vee$  ( $i = 0, 1$ ) are invertible with  $t_i^{-1} = c_i - t_i$  and  $t_i^{\vee-1} = c_i^\vee - t_i^\vee$ . Set

$$u = t_1 t_0^\vee, \quad v = t_1^\vee t_1.$$

Clearly  $t_0^\vee = t_1^{-1}u$  and  $t_1^\vee = vt_1^{-1}$ . Relation (27) implies that  $t_0 = q^{-1}u^{-1}t_1v^{-1}$ . Therefore  $t_1^{\pm 1}, u^{\pm 1}, v^{\pm 1}$  form a set of generators of  $\mathfrak{H}$ . Furthermore [22, Proposition 5.2] implies that

**Proposition 5.1.** *The  $\mathbb{F}$ -algebra  $\mathfrak{H}$  is generated by  $t_1^{\pm 1}, u^{\pm 1}, v^{\pm 1}$  subject to the relations  $t_1 t_1^{-1} = t_1^{-1} t_1 = 1, uu^{-1} = u^{-1}u = 1, vv^{-1} = v^{-1}v = 1$  and*

$$\begin{aligned} t_1^{-1} &= c_1 - t_1, \\ ut_1^{-1} &= -t_1 u^{-1} + c_0^\vee, \\ vt_1^{-1} &= -t_1 v^{-1} + c_1^\vee, \\ v^{-1}u^{-1} &= q^2 u^{-1} v^{-1} + q^2 c_1 t_1^{-1} uv^{-1} - q^2 c_1^\vee t_1^{-1} u + c_0^\vee (q^2 t_1 - q^{-2} c_1) v^{-1} + q c_0 t_1^{-1}, \\ vu &= q^2 uv + q^{-2} c_1 t_1^{-1} uv^{-1} - c_1^\vee t_1^{-1} u + c_0^\vee (t_1 - q^{-2} c_1) v^{-1} - q c_0 (t_1 - q^{-2} c_1), \\ v^{-1}u &= -q^{-2} t_1^{-2} uv^{-1} + c_1^\vee t_1^{-1} u + q^{-2} c_0^\vee t_1^{-1} v^{-1} - q^{-1} c_0 t_1^{-1}, \\ vu^{-1} &= q^{-2} u^{-1} v - q^{-2} c_1 t_1^{-1} uv^{-1} + q^{-2} c_1^\vee t_1^{-1} u + c_0^\vee (q^{-2} c_1 - t_1) v^{-1} - q^{-1} c_0 t_1^{-1} \\ &\quad + (1 - q^{-2}) c_0^\vee c_1^\vee. \end{aligned}$$

Applying the Bergman's diamond lemma [2, Theorem 1.2] to Proposition 5.1, it follows that

**Lemma 5.2.**  $\mathfrak{H}$  has the  $\mathbb{F}$ -basis

$$t_1^\ell u^i v^j \quad \text{for all } \ell \in \{0, 1\} \text{ and } i, j \in \mathbb{Z}.$$

Let  $\mathbb{T}$  denote the  $\mathbb{F}$ -subalgebra of  $\mathfrak{H}$  generated by  $t_1$ .

**Lemma 5.3.**  $\mathbb{T}$  is the  $\mathbb{F}$ -algebra generated by  $t_1^{\pm 1}$  subject to the relations

$$t_1 t_1^{-1} = t_1^{-1} t_1 = 1, \quad t_1 + t_1^{-1} = c_1.$$

*Proof.* Apply the first and fourth relations in Proposition 5.1. □

To make the arguments brief, Lemma 5.3 will be used tacitly henceforward. For each  $n \in \mathbb{N}$  denote by  $\mathfrak{H}_n$  the left  $\mathbb{T}$ -submodule of  $\mathfrak{H}$  spanned by

$$u^i v^j \quad \text{for all } i, j \in \mathbb{Z} \text{ with } |i| + |j| \leq n.$$

As a left  $\mathbb{T}$ -module,  $\mathfrak{H}$  has the basis

$$u^i v^j \quad \text{for all } i, j \in \mathbb{Z}$$

by Lemmas 5.2. Thus  $\mathfrak{H} = \bigcup_{n \in \mathbb{N}} \mathfrak{H}_n$ . For any  $m, n \in \mathbb{N}$ ,  $\mathfrak{H}_m \cdot \mathfrak{H}_n \subseteq \mathfrak{H}_{m+n}$  by Proposition 5.1. Therefore the increasing sequence

$$\mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \cdots \subseteq \mathfrak{H}_n \subseteq \cdots$$

gives an  $\mathbb{N}$ -filtration of the  $\mathbb{T}$ -algebra  $\mathfrak{H}$ . From the relations in Proposition 5.1 we see that

$$(28) \quad ut_1^{-1} = -t_1 u^{-1} \pmod{\mathfrak{H}_0},$$

$$(29) \quad vt_1^{-1} = -t_1 v^{-1} \pmod{\mathfrak{H}_0},$$

$$(30) \quad v^{-1}u^{-1} = q^2 u^{-1} v^{-1} + q^2 c_1 t_1^{-1} uv^{-1} \pmod{\mathfrak{H}_1},$$

$$(31) \quad v^{-1}u = -q^{-2} t_1^{-2} uv^{-1} \pmod{\mathfrak{H}_1},$$

$$(32) \quad vu^{-1} = q^{-2} u^{-1} v - q^{-2} c_1 t_1^{-1} uv^{-1} \pmod{\mathfrak{H}_1}.$$

**5.2. The centralizer of  $t_1$  in  $\mathfrak{H}$ .** By [22, §6] or [35, §16] there exists a unique  $\mathbb{F}$ -algebra homomorphism  $\sharp : \Delta \rightarrow \mathfrak{H}$  given by

$$\begin{aligned} A^\sharp &= t_1 t_0^\vee + (t_1 t_0^\vee)^{-1}, \\ B^\sharp &= t_1^\vee t_1 + (t_1^\vee t_1)^{-1}, \\ C^\sharp &= t_0 t_1 + (t_0 t_1)^{-1}, \\ \alpha^\sharp &= \phi_1^0(c_0^\vee, c_1^\vee, c_0; q^{-1}t_1 + qt_1^{-1}), \\ \beta^\sharp &= \phi_1^1(c_0^\vee, c_1^\vee, c_0; q^{-1}t_1 + qt_1^{-1}), \\ \gamma^\sharp &= \phi_1^2(c_0^\vee, c_1^\vee, c_0; q^{-1}t_1 + qt_1^{-1}), \\ \Omega^\sharp &= \psi_1(c_0^\vee, c_1^\vee, c_0; q^{-1}t_1 + qt_1^{-1}). \end{aligned}$$

The image of  $\sharp$  is contained in the centralizer  $C_{\mathfrak{H}}(t_1)$  of  $t_1$  in  $\mathfrak{H}$  [18, Theorem 2.4(i)]. For convenience we denote by  $A, B, C$  the elements

$$t_1 t_0^\vee + (t_1 t_0^\vee)^{-1}, \quad t_1^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad t_0 t_1 + (t_0 t_1)^{-1}$$

respectively. In this subsection we shall show that

**Theorem 5.4.** *The elements*

$$A^i C^j B^k \quad \text{for all } i, j, k \in \mathbb{N} \text{ with } ijk = 0$$

*form a  $\mathbb{T}$ -basis of  $C_{\mathfrak{H}}(t_1)$ .*

Before delving into the proof of Theorem 5.4 we lay some groundwork. Applying (28), (29) the routine inductions yield that

**Lemma 5.5.** *For each  $i \in \mathbb{N}^*$*

$$\begin{aligned} u^i t_1^{-1} &= -t_1 u^{-i} \pmod{\mathfrak{H}_{i-1}}, \\ v^i t_1^{-1} &= -t_1 v^{-i} \pmod{\mathfrak{H}_{i-1}}. \end{aligned}$$

As consequences of Lemma 5.5 we have

**Lemma 5.6.** *For all  $i, j \in \mathbb{N}^*$*

$$\begin{aligned} u^i v^j t_1^{-1} &= -t_1 u^{-i} v^{-j} - c_1 u^i v^{-j} \pmod{\mathfrak{H}_{i+j-1}}, \\ u^{-i} v^j t_1^{-1} &= t_1^{-1} u^i v^{-j} \pmod{\mathfrak{H}_{i+j-1}}, \\ u^{-i} v^{-j} t_1 &= -t_1^{-1} u^i v^j - c_1 u^{-i} v^j \pmod{\mathfrak{H}_{i+j-1}}. \end{aligned}$$

**Proposition 5.7.** *For each  $i \in \mathbb{N}^*$*

$$\begin{aligned} A^i &= u^i + u^{-i} \pmod{\mathfrak{H}_{i-1}}, \\ B^i &= v^i + v^{-i} \pmod{\mathfrak{H}_{i-1}}, \\ C^i &= (-1)^i q^{-i^2} (u^{-i} v^i - t_1^{-2} u^i v^{-i}) \pmod{\mathfrak{H}_{2i-1}}. \end{aligned}$$

*Proof.* By construction

$$A = u + u^{-1}, \quad B = v + v^{-1}.$$

The equalities for  $A^i, B^i \pmod{\mathfrak{H}_{i-1}}$  are immediate from the binomial theorem. A direct calculation yields that

$$C = q^{-1} t_1^{-2} u v^{-1} - q^{-1} u^{-1} v - q^{-1} c_1^\vee t_1^{-1} u - q^{-1} c_0^\vee t_1^{-1} v^{-1} + c_0 t_1^{-1} + q^{-1} c_0^\vee c_1^\vee$$

as a linear combination of the basis of  $\mathfrak{H}$  given in Lemma 5.2. The expression implies that

$$C^i = q^{-i} (r - s)^i \pmod{\mathfrak{H}_{2i-1}}$$

where  $r = t_1^{-2}uv^{-1}$  and  $s = u^{-1}v$ . Equation (30) implies that

$$rs \in \mathfrak{H}_3.$$

Decompose the product  $sr$  into  $(u^{-1}vt_1^{-1})(t_1^{-1}uv^{-1})$ . The second equation in Lemma 5.6 with  $(i, j) = (1, 1)$  says that

$$(33) \quad u^{-1}vt_1^{-1} = t_1^{-1}uv^{-1} \pmod{\mathfrak{H}_1}.$$

Therefore

$$sr = t_1^{-1}uv^{-1}u^{-1}vt_1^{-1} \pmod{\mathfrak{H}_3}.$$

By (30) the right-hand side is equal to 0 mod  $\mathfrak{H}_3$ . Therefore

$$sr \in \mathfrak{H}_3.$$

Since  $r$  and  $s$  are in  $\mathfrak{H}_2$ , the relations  $rs, sr \in \mathfrak{H}_3$  imply that  $(r - s)^i = r^i + (-1)^i s^i \pmod{\mathfrak{H}_{2i-1}}$ . Applying (32) a routine induction shows that

$$s^i = q^{-i(i-1)}u^{-i}v^i \pmod{\mathfrak{H}_{2i-1}}.$$

To see the equality for  $C^i \pmod{\mathfrak{H}_{2i-1}}$  we have to show that

$$(34) \quad r^i = (-1)^{i-1}q^{-i(i-1)}t_1^{-2}u^i v^{-i} \pmod{\mathfrak{H}_{2i-1}}.$$

Proceed by induction on  $i$ . Applying the induction hypothesis

$$r^i = r \cdot r^{i-1} = (-1)^i q^{-(i-1)(i-2)} t_1^{-2} uv^{-1} t_1^{-2} u^{i-1} v^{1-i} \pmod{\mathfrak{H}_{2i-1}}$$

for  $i \geq 2$ . Equation (33) can be written as  $uv^{-1}t_1^{-1} = c_1 uv^{-1} - t_1 u^{-1}v \pmod{\mathfrak{H}_1}$ . Therefore  $r^i \pmod{\mathfrak{H}_{2i-1}}$  is equal to  $(-1)^i q^{-(i-1)(i-2)}$  times

$$(35) \quad c_1 t_1^{-2} uv^{-1} t_1^{-1} u^{i-1} v^{1-i} - t_1^{-1} u^{-1} v t_1^{-1} u^{i-1} v^{1-i}.$$

Consider the element  $w = uv^{-1}t_1^{-1}u$  in the minuend of (35). Equation (28) implies that  $w = -uv^{-1}u^{-1}t_1 \pmod{\mathfrak{H}_2}$ . Followed by using equation (31) it yields that

$$w = q^2 t_1^2 v^{-1} t_1 \pmod{\mathfrak{H}_2}.$$

This shows  $w \in \mathfrak{H}_2$  and thus the minuend of (35) lies in  $\mathfrak{H}_{2i-1}$ . Therefore  $r^i \pmod{\mathfrak{H}_{2i-1}}$  is simplified to be

$$(-1)^{i-1} q^{-(i-1)(i-2)} t_1^{-1} u^{-1} v t_1^{-1} u^{i-1} v^{1-i}.$$

By (33) it follows that

$$r^i = (-1)^{i-1} q^{-(i-1)(i-2)} t_1^{-2} uv^{-1} u^{i-1} v^{1-i} \pmod{\mathfrak{H}_{2i-1}}.$$

To see (34) it remains to prove that

$$uv^{-1}u^{i-1} = q^{-2(i-1)}u^i v^{-1} \pmod{\mathfrak{H}_i}.$$

The equation follows by applying the case  $i = 2$  to a routine induction. By left multiplication by  $u$  on (31) we have

$$uv^{-1}u = -q^{-2}ut_1^{-2}uv^{-1}u \pmod{\mathfrak{H}_2}.$$

Applying (28) twice, one may find that

$$ut_1^{-2}u = -u^2 \pmod{\mathfrak{H}_1}.$$

The case  $i = 2$  now follows by combining the above two equations. This proposition follows.  $\square$

It is ready to prove Theorem 5.4.

*Proof of Theorem 5.4.* Proceed by induction on  $n \in \mathbb{N}$  to show that  $C_{\mathfrak{H}}(t_1) \cap \mathfrak{H}_n$  has the  $\mathbb{T}$ -basis

$$A^i C^j B^k \quad \text{for all } i, j, k \in \mathbb{N} \text{ with } i + 2j + k \leq n \text{ and } ijk = 0.$$

There is nothing to prove for  $n = 0$ . Suppose that  $n \geq 1$ . By induction hypothesis it suffices to show that for any  $R \in C_{\mathfrak{H}}(t_1) \cap \mathfrak{H}_n$  the equation

$$(36) \quad R = \sum_{\substack{i,j \in \mathbb{N} \\ i+j=n}} s_{i,j} A^i B^j + \sum_{\substack{i \in \mathbb{N}, j \in \mathbb{N}^* \\ i+2j=n}} t_{i,j} A^i C^j + \sum_{\substack{i,j \in \mathbb{N}^* \\ i+2j=n}} u_{i,j} C^j B^i \quad (\text{mod } \mathfrak{H}_{n-1})$$

has a unique solution for  $s_{i,j}, t_{i,j}, u_{i,j} \in \mathbb{T}$ . By Lemma 5.2 there are unique  $r_{i,j} \in \mathbb{T}$  for all  $i, j \in \mathbb{Z}$  with  $|i| + |j| = n$  such that

$$R = \sum_{\substack{i,j \in \mathbb{Z} \\ |i|+|j|=n}} r_{i,j} u^i v^j \quad (\text{mod } \mathfrak{H}_{n-1}).$$

Apply Lemmas 5.5 and 5.6 to evaluate the coefficients of  $u^i v^j$  in  $[R, t_1]$  for all  $i, j \in \mathbb{Z}$  with  $|i| + |j| = n$ . On the other hand, since  $R \in C_{\mathfrak{H}}(t_1)$  the commutator  $[R, t_1] = 0$  and it follows that

$$r_{i,j} = r_{-i,-j}, \quad c_1 r_{i,j} = t_1 r_{i,-j} + t_1^{-1} r_{-i,j}$$

for all  $i, j \in \mathbb{N}$  with  $i + j = n$ . As a result,  $R \text{ mod } \mathfrak{H}_{n-1}$  is equal to

$$r_{n,0}(u^n + u^{-n}) + r_{0,n}(v^n + v^{-n}) + \sum_{\substack{i,j \in \mathbb{N}^* \\ i+j=n}} r_{i,j}(u^i + u^{-i})(v^j + v^{-j}) + (r_{i,j} - r_{i,-j})(t_1^2 u^{-i} v^j - u^i v^{-j}).$$

By Proposition 5.7 we have

$$\begin{aligned} A^n &= u^n + u^{-n} & (\text{mod } \mathfrak{H}_{n-1}), \\ B^n &= v^n + v^{-n} & (\text{mod } \mathfrak{H}_{n-1}), \end{aligned}$$

and a direct calculation yields that

$$A^i B^j = (u^i + u^{-i})(v^j + v^{-j}) \quad (\text{mod } \mathfrak{H}_{n-1})$$

for all  $i, j \in \mathbb{N}^*$  with  $i + j = n$ ,

$$A^i C^j = (-1)^j q^{-j^2} (u^{-i-j} v^j - t_1^{-2} u^{i+j} v^{-j}) \quad (\text{mod } \mathfrak{H}_{n-1})$$

for all  $i \in \mathbb{N}, j \in \mathbb{N}^*$  with  $i + 2j = n$  and

$$C^j B^i = (-1)^j q^{-j^2} (u^{-j} v^{i+j} - t_1^{-2} u^j v^{-i-j}) \quad (\text{mod } \mathfrak{H}_{n-1})$$

for all  $i, j \in \mathbb{N}^*$  with  $i + 2j = n$ . The comparison of coefficients implies that equation (36) has the following unique solution

$$\begin{aligned} s_{i,j} &= r_{i,j}, \\ t_{i,j} &= (-1)^j q^{j^2} t_1^2 (r_{i+j,j} - r_{i+j,-j}), \\ u_{i,j} &= (-1)^j q^{j^2} t_1^2 (r_{j,i+j} - r_{j,-i-j}), \end{aligned}$$

as desired. □

**5.3. A presentation for  $Z(\mathfrak{H})$ .** Now the ideas from §3 and §4 can be used to give a presentation for  $Z(\mathfrak{H})$ .

**Theorem 5.8.** *For any nonzero multiple  $n$  of  $\bar{d}$  the elements*

$$T_n(A) = (t_1 t_0^\vee)^n + (t_1 t_0^\vee)^{-n}, \quad T_n(B) = (t_1^\vee t_1)^n + (t_1^\vee t_1)^{-n}, \quad T_n(C) = (t_0 t_1)^n + (t_0 t_1)^{-n}$$

*are central in  $\mathfrak{H}$ .*

*Proof.* By [28, Proposition 2.4] there exists an  $\mathbb{F}$ -algebra isomorphism from  $\mathfrak{H} = \mathfrak{H}_q(k_0, k_1, k_0^\vee, k_1^\vee)$  into  $\mathfrak{H}_q(k_1^\vee, k_1, k_0, k_0^\vee)$  that sends

$$(t_0, t_1, t_0^\vee, t_1^\vee) \mapsto (t_0^\vee, t_1, t_1^{-1} t_1^\vee t_1, t_0),$$

where  $t_0, t_1, t_0^\vee, t_1^\vee$  on the right-hand side denote the defining generators of  $\mathfrak{H}_q(k_1^\vee, k_1, k_0, k_0^\vee)$  with the relations (26) in which  $k_0, k_1, k_0^\vee, k_1^\vee$  are replaced by  $k_1^\vee, k_1, k_0, k_0^\vee$  respectively. A direct calculation shows that the isomorphism sends  $A, B, C \in \mathfrak{H}$  to the corresponding  $B, C, A \in \mathfrak{H}_q(k_1^\vee, k_1, k_0, k_0^\vee)$  respectively. Since  $k_0, k_1, k_0^\vee, k_1^\vee$  are arbitrary nonzero scalars in  $\mathbb{F}$ , it suffices to show that  $T_n(B)$  is central in  $\mathfrak{H}$  when  $\bar{d}$  divides  $n$ .

As mentioned before, the element  $B \in C_{\mathfrak{H}}(t_1)$ . Clearly  $B = v + v^{-1}$  commutes with  $v$ . Since the  $\mathbb{F}$ -algebra  $\mathfrak{H}$  is generated by  $t_1^{\pm 1}, u^{\pm 1}, v^{\pm 1}$  by Proposition 5.1, it is enough to verify that  $T_n(B) = v^n + v^{-n}$  commutes with  $u$  when  $\bar{d}$  divides  $n$ . To see this express  $B^n u$  ( $n \in \mathbb{N}$ ) as a linear combination of the  $\mathbb{T}$ -basis of  $\mathfrak{H}$  given below Lemma 5.3:

$$B^n u = u(q^2 v + q^{-2} v^{-1})^n + t_1(q^{-1} c_0^\vee v^{-1} - c_0) \frac{(q^2 v + q^{-2} v^{-1})^n - (v + v^{-1})^n}{qv - q^{-1} v^{-1}}.$$

Using this identity the element  $T_n(B)u$  is equal to

$$uT_n(q^2 v + q^{-2} v^{-1}) + t_1(q^{-1} c_0^\vee v^{-1} - c_0) \frac{T_n(q^2 v + q^{-2} v^{-1}) - T_n(v + v^{-1})}{qv - q^{-1} v^{-1}}.$$

Simplifying the above by using Lemma 2.10(i) it becomes

$$u(q^{2n} v^n + q^{-2n} v^{-n}) + t_1(q^n - q^{-n})(q^{-1} c_0^\vee v^{-1} - c_0) \frac{q^n v^n - q^{-n} v^{-n}}{qv - q^{-1} v^{-1}}.$$

Since  $q^{2\bar{d}} = 1$  the above is equal to  $uT_n(B)$  when  $\bar{d}$  divides  $n$ , as desired.  $\square$

**Theorem 5.9.** *The elements*

$$T_{\bar{d}}(A)^i T_{\bar{d}}(B)^j T_{\bar{d}}(C)^k \quad \text{for all } i, j, k \in \mathbb{N} \text{ with } ijk = 0$$

*form an  $\mathbb{F}$ -basis of  $Z(\mathfrak{H})$ .*

*Proof.* Inspired by Theorem 5.4 we consider the  $\mathbb{N}$ -filtration

$$C_{\mathfrak{H}}(t_1)_0 \subseteq C_{\mathfrak{H}}(t_1)_1 \subseteq \cdots \subseteq C_{\mathfrak{H}}(t_1)_n \subseteq \cdots$$

of the  $\mathbb{T}$ -algebra  $C_{\mathfrak{H}}(t_1)$ , where  $C_{\mathfrak{H}}(t_1)_n$  is defined to be the  $\mathbb{T}$ -submodule of  $C_{\mathfrak{H}}(t_1)$  spanned by

$$A^i C^j B^k \quad \text{for all } i, j, k \in \mathbb{N} \text{ with } ijk = 0 \text{ and } i + j + k \leq n$$

rather than  $C_{\mathfrak{H}}(t_1) \cap \mathfrak{H}_n$ . Applying the  $\mathbb{N}$ -filtration of  $C_{\mathfrak{H}}(t_1)$  and by Theorems 5.4 and 5.8, a similar argument to the proof of Theorem 4.1 show that  $Z(\mathfrak{H})$  has the  $\mathbb{T}$ -basis

$$T_{\bar{d}}(A)^i T_{\bar{d}}(B)^j T_{\bar{d}}(C)^k \quad \text{for all } i, j, k \in \mathbb{N} \text{ with } ijk = 0.$$

Therefore any element  $R \in Z(\mathfrak{H})$  can be uniquely expressed as a linear combination

$$\sum_{\substack{i, j, k \in \mathbb{N} \\ ijk = 0}} r_{i, j, k} T_{\bar{d}}(A)^i T_{\bar{d}}(B)^j T_{\bar{d}}(C)^k$$



where the coefficients  $r_{i,j,k} \in \mathbb{T}$ . It remains to prove that the coefficients  $r_{i,j,k} \in \mathbb{F}$ . To do this we invoke the fact:

**Lemma 5.10.**  $\mathbb{T} \cap Z(\mathfrak{H}) = \mathbb{F}$ .

*Proof.* By (28) the commutator  $[t_1^{-1}, u] = t_1^{-1}u + t_1u^{-1} \bmod \mathfrak{H}_0$ . Therefore  $[t_1^{-1}, u] \neq 0 \bmod \mathfrak{H}_0$  by Lemma 5.2. This shows that  $t_1^{-1} \notin Z(\mathfrak{H})$ . Since  $\{1, t_1^{-1}\}$  is an  $\mathbb{F}$ -basis of  $\mathbb{T}$  this lemma follows.  $\square$

Since  $R$  is central in  $\mathfrak{H}$  the commutator  $[R, S] = 0$  for any  $S \in \mathfrak{H}$ . On the other hand, Theorem 5.8 implies that  $[R, S]$  is equal to

$$\sum_{\substack{i,j,k \in \mathbb{N} \\ i+j+k=0}} [r_{i,j,k}, S] T_d(A)^i T_d(B)^j T_d(C)^k.$$

This forces that  $[r_{i,j,k}, S] = 0$  and thus  $r_{i,j,k} \in Z(\mathfrak{H})$ . By Lemma 5.10 the coefficients  $r_{i,j,k} \in \mathbb{F}$  as claimed.  $\square$

The Oblomkov presentation for  $Z(\mathfrak{H})$  at  $q = 1$  [28, Theorem 3.1] now can be generalized to any root of unity  $q$ .

**Theorem 5.11.** *Let*

$$A = t_1 t_0^\vee + (t_1 t_0^\vee)^{-1}, \quad B = t_1^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad C = t_0 t_1 + (t_0 t_1)^{-1}.$$

*Let  $c_0, \tilde{c}_1, c_0^\vee, c_1^\vee$  denote the scalars*

$$k_0 + k_0^{-1}, \quad q^{-1}k_1 + qk_1^{-1}, \quad k_0^\vee + k_0^{\vee-1}, \quad k_1^\vee + k_1^{\vee-1}$$

*respectively. Then  $Z(\mathfrak{H})$  is the commutative  $\mathbb{F}$ -algebra generated by  $T_d(A), T_d(B), T_d(C)$  subject to the relation*

$$(37) \quad \begin{aligned} & q^{\tilde{d}} \phi_d^0(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) T_d(A) + q^{\tilde{d}} \phi_d^1(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) T_d(B) + q^{\tilde{d}} \phi_d^2(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) T_d(C) \\ &= q^{\tilde{d}} T_d(A) T_d(B) T_d(C) + T_d(A)^2 + T_d(B)^2 + T_d(C)^2 + \psi_d(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) - 2. \end{aligned}$$

*Proof.* As a consequence of Theorem 5.9 the center of  $\mathfrak{H}$  is generated by  $T_d(A), T_d(B), T_d(C)$ . By Lemma 2.10(i), for each  $n \in \mathbb{N}$

$$T_n(q^{-1}t_1 + qt_1^{-1}) = q^{-n}t_1^n + q^n t_1^{-n}, \quad T_n(\tilde{c}_1) = q^{-n}k_1^n + q^n k_1^{-n}.$$

Both are equal to  $q^n T_n(c_1)$  when  $\tilde{d}$  divides  $n$ . Recall the homomorphism  $\sharp : \Delta \rightarrow \mathfrak{H}$  from Theorem 5.4 above. Applying Theorem 4.5 it follows that

$$\Phi_d(\alpha, \beta, \gamma; \Omega)^\sharp = \phi_d(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1), \quad \Psi_d(\alpha, \beta, \gamma; \Omega)^\sharp = \psi_d(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1).$$

Therefore (37) holds by Theorem 4.6.

Let  $K$  denote the  $\mathbb{F}$ -subalgebra of  $Z(\mathfrak{H})$  generated by  $T_d(A), T_d(B)$ . Relation (37) implies that  $T_d(C)$  is a root of the quadratic polynomial

$$\begin{aligned} P(X) &= X^2 + q^{\tilde{d}} (T_d(A) T_d(B) - \phi_d^2(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1)) X + T_d(A)^2 + T_d(B)^2 \\ &\quad - q^{\tilde{d}} \phi_d^0(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) T_d(A) - q^{\tilde{d}} \phi_d^1(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) T_d(B) + \psi_d(c_0^\vee, c_1^\vee, c_0; \tilde{c}_1) - 2. \end{aligned}$$

By Theorem 5.9,  $K$  is a two-variable polynomial ring over  $\mathbb{F}$  generated by  $T_d(A), T_d(B)$ . In particular  $K$  is a unique factorization domain. Observe that  $P(X)$  is irreducible over  $K$ . By Gauss's Lemma  $P(X)$  is the minimal polynomial of  $T_d(C)$  over the fraction field of  $K$ . This shows the theorem.  $\square$

## APPENDIX. AUXILIARY DATA

Assume that  $q^2 \neq 1$ . As the linear combinations of the  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}[\Lambda]$ -basis (24) of  $\mathbb{F}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] \otimes_{\mathbb{F}} Z(U)$ , the nonzero coefficients of

$$\begin{aligned} & T_d(A)^{\natural}, \quad T_d(B)^{\natural}, \quad T_d(C)^{\natural}, \quad T_d(A)^{2\natural}, \quad T_d(B)^{2\natural}, \quad T_d(C)^{2\natural}, \\ & T_d(A)^{\natural}T_d(B)^{\natural}, \quad T_d(B)^{\natural}T_d(C)^{\natural}, \quad T_d(C)^{\natural}T_d(A)^{\natural}, \quad T_d(A)^{\natural}T_d(B)^{\natural}T_d(C)^{\natural} \end{aligned}$$

can be evaluated by applying relation (10) to Theorem 3.9. The results are listed as follows:

| coefficients               |          | $T_d(A)^{\natural}$                               | f | coefficients               |          | $T_d(B)^{\natural}$  |
|----------------------------|----------|---|---|----------------------------|----------|--|
| $k^{\vec{d}i}$             | $i = -1$ | $a^{\vec{d}}$                                     |   | $k^{\vec{d}i} f^{\vec{d}}$ | $i = 0$  | $(q - q^{-1})^{\vec{d}} b^{-\vec{d}}$                          |
|                            | $i = 1$  | $a^{-\vec{d}}$                                    |   |                            | $i = 1$  | $-(q - q^{-1})^{\vec{d}} q^{\vec{d}} a^{-\vec{d}} c^{\vec{d}}$ |
| $k^{\vec{d}i} e^{\vec{d}}$ | $i = -1$ | $-(q - q^{-1})^{\vec{d}} q^{\vec{d}} a^{\vec{d}}$ |   | $k^{\vec{d}i}$             | $i = -1$ | $b^{-\vec{d}}$   |
|                            | $i = 0$  | $(q - q^{-1})^{\vec{d}} b^{\vec{d}} c^{-\vec{d}}$ |   |                            | $i = 1$  | $b^{\vec{d}}$  |

| coefficients               |          | $T_d(C)^{\natural}$  |
|----------------------------|----------|--|
| $k^{\vec{d}i} f^{\vec{d}}$ | $i = -1$ | $-(q - q^{-1})^{\vec{d}} q^{\vec{d}} a^{\vec{d}} b^{-\vec{d}}$       |
|                            | $i = 0$  | $(q - q^{-1})^{\vec{d}} c^{\vec{d}}$                                 |
| $k^{\vec{d}i}$             | $i = -2$ | $-2q^{\vec{d}} a^{\vec{d}} b^{-\vec{d}}$                             |
|                            | $i = -1$ | $a^{\vec{d}} b^{-\vec{d}} T_d(\Lambda) + c^{\vec{d}} + c^{-\vec{d}}$ |
| $k^{\vec{d}i} e^{\vec{d}}$ | $i = -2$ | $(q - q^{-1})^{\vec{d}} a^{\vec{d}} b^{-\vec{d}}$                    |
|                            | $i = -1$ | $-(q - q^{-1})^{\vec{d}} q^{\vec{d}} c^{-\vec{d}}$                   |

| coefficients                |          | $T_d(A)^{2\natural}$   |  | coefficients                |          | $T_d(B)^{2\natural}$  |
|-----------------------------|----------|--|--|-----------------------------|----------|---|
| $k^{\vec{d}i}$              | $i = -2$ | $a^{2\vec{d}}$   |  | $k^{\vec{d}i} f^{2\vec{d}}$ | $i = 0$  | $(q - q^{-1})^{2\vec{d}} b^{-2\vec{d}}$                                       |
|                             | $i = 0$  | 2  |  |                             | $i = 1$  | $-2(q - q^{-1})^{2\vec{d}} q^{\vec{d}} a^{-\vec{d}} b^{-\vec{d}} c^{\vec{d}}$ |
|                             | $i = 2$  | $a^{-2\vec{d}}$  |  |                             | $i = 2$  | $(q - q^{-1})^{2\vec{d}} a^{-2\vec{d}} c^{2\vec{d}}$                          |
| $k^{\vec{d}i} e^{\vec{d}}$  | $i = -2$ | $-2(q - q^{-1})^{\vec{d}} q^{\vec{d}} a^{2\vec{d}}$                          |  | $k^{\vec{d}i} f^{\vec{d}}$  | $i = -1$ | $2(q - q^{-1})^{\vec{d}} b^{-2\vec{d}}$                                       |
|                             | $i = -1$ | $2(q - q^{-1})^{\vec{d}} a^{\vec{d}} b^{\vec{d}} c^{-\vec{d}}$               |  |                             | $i = 0$  | $-2(q - q^{-1})^{\vec{d}} q^{\vec{d}} a^{-\vec{d}} b^{-\vec{d}} c^{\vec{d}}$  |
|                             | $i = 0$  | $-2(q - q^{-1})^{\vec{d}} q^{\vec{d}}$                                       |  |                             | $i = 1$  | $2(q - q^{-1})^{\vec{d}}$   |
|                             | $i = 1$  | $2(q - q^{-1})^{\vec{d}} a^{-\vec{d}} b^{\vec{d}} c^{-\vec{d}}$              |  |                             | $i = 2$  | $-2(q - q^{-1})^{\vec{d}} q^{\vec{d}} a^{-\vec{d}} b^{\vec{d}} c^{\vec{d}}$   |
| $k^{\vec{d}i} e^{2\vec{d}}$ | $i = -2$ | $(q - q^{-1})^{2\vec{d}} a^{2\vec{d}}$                                       |  | $k^{\vec{d}i}$              | $i = -2$ | $b^{-2\vec{d}}$   |
|                             | $i = -1$ | $-2(q - q^{-1})^{2\vec{d}} q^{\vec{d}} a^{\vec{d}} b^{\vec{d}} c^{-\vec{d}}$ |  |                             | $i = 0$  | 2   |
|                             | $i = 0$  | $(q - q^{-1})^{2\vec{d}} b^{2\vec{d}} c^{-2\vec{d}}$                         |  |                             | $i = 2$  | $b^{2\vec{d}}$  |

| coefficients                          |          | $T_d(C)^{2\mathfrak{d}}$   |
|---------------------------------------|----------|--|
| $k^{\mathfrak{d}i} f^{2\mathfrak{d}}$ | $i = -2$ | $(q - q^{-1})^{2\mathfrak{d}} a^{2\mathfrak{d}} b^{-2\mathfrak{d}}$  |
|                                       | $i = -1$ | $-2(q - q^{-1})^{2\mathfrak{d}} q^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}} c^{\mathfrak{d}}$  |
|                                       | $i = 0$  | $(q - q^{-1})^{2\mathfrak{d}} c^{2\mathfrak{d}}$   |
| $k^{\mathfrak{d}i} f^{\mathfrak{d}}$  | $i = -3$ | $4(q - q^{-1})^{\mathfrak{d}} a^{2\mathfrak{d}} b^{-2\mathfrak{d}}$  |
|                                       | $i = -2$ | $-2(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}} (a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + 3c^{\mathfrak{d}} + c^{-\mathfrak{d}})$                      |
|                                       | $i = -1$ | $2(q - q^{-1})^{\mathfrak{d}} c^{\mathfrak{d}} (a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}})$   |
| $k^{\mathfrak{d}i}$                   | $i = -4$ | $6a^{2\mathfrak{d}} b^{-2\mathfrak{d}}$  |
|                                       | $i = -3$ | $-6q^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}} (a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}})$   |
|                                       | $i = -2$ | $a^{2\mathfrak{d}} b^{-2\mathfrak{d}} (T_d(\Lambda)^2 + 2) + (c^{\mathfrak{d}} + c^{-\mathfrak{d}}) (4a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}}) + 2$ |
|                                       | $i = -1$ | $-2q^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}} (a^{-\mathfrak{d}} b^{\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}})$   |
|                                       | $i = 0$  | $2$  |
| $k^{\mathfrak{d}i} e^{\mathfrak{d}}$  | $i = -4$ | $-4(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} a^{2\mathfrak{d}} b^{-2\mathfrak{d}}$  |
|                                       | $i = -3$ | $2(q - q^{-1})^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}} (a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + 3c^{-\mathfrak{d}})$  |
|                                       | $i = -2$ | $-2(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} c^{-\mathfrak{d}} (a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}})$  |
| $k^{\mathfrak{d}i} e^{2\mathfrak{d}}$ | $i = -4$ | $(q - q^{-1})^{2\mathfrak{d}} a^{2\mathfrak{d}} b^{-2\mathfrak{d}}$  |
|                                       | $i = -3$ | $-2(q - q^{-1})^{2\mathfrak{d}} q^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}} c^{-\mathfrak{d}}$   |
|                                       | $i = -2$ | $(q - q^{-1})^{2\mathfrak{d}} c^{-2\mathfrak{d}}$  |

| coefficients                         |          | $T_d(A)^{\mathfrak{d}} T_d(B)^{\mathfrak{d}}$   |
|--------------------------------------|----------|---|
| $k^{\mathfrak{d}i} f^{\mathfrak{d}}$ | $i = -1$ | $(q - q^{-1})^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}}$  |
|                                      | $i = 0$  | $-(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} c^{\mathfrak{d}}$  |
|                                      | $i = 1$  | $(q - q^{-1})^{\mathfrak{d}} a^{-\mathfrak{d}} b^{-\mathfrak{d}}$   |
|                                      | $i = 2$  | $-(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} a^{-2\mathfrak{d}} c^{\mathfrak{d}}$   |
| $k^{\mathfrak{d}i}$                  | $i = -2$ | $2a^{\mathfrak{d}} b^{-\mathfrak{d}}$   |
|                                      | $i = -1$ | $-q^{\mathfrak{d}} (a^{\mathfrak{d}} b^{-\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}})$                          |
|                                      | $i = 0$  | $(c^{\mathfrak{d}} + c^{-\mathfrak{d}}) T_d(\Lambda) + (a^{\mathfrak{d}} + a^{-\mathfrak{d}}) (b^{\mathfrak{d}} + b^{-\mathfrak{d}})$ |
|                                      | $i = 1$  | $-q^{\mathfrak{d}} (a^{-\mathfrak{d}} b^{\mathfrak{d}} T_d(\Lambda) + c^{\mathfrak{d}} + c^{-\mathfrak{d}})$                          |
|                                      | $i = 2$  | $2a^{-\mathfrak{d}} b^{\mathfrak{d}}$   |
| $k^{\mathfrak{d}i} e^{\mathfrak{d}}$ | $i = -2$ | $-(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} a^{\mathfrak{d}} b^{-\mathfrak{d}}$  |
|                                      | $i = -1$ | $(q - q^{-1})^{\mathfrak{d}} c^{-\mathfrak{d}}$   |
|                                      | $i = 0$  | $-(q - q^{-1})^{\mathfrak{d}} q^{\mathfrak{d}} a^{\mathfrak{d}} b^{\mathfrak{d}}$   |
|                                      | $i = 1$  | $(q - q^{-1})^{\mathfrak{d}} b^{2\mathfrak{d}} c^{-\mathfrak{d}}$   |

| coefficients               |          | $T_d(B)^{\natural}T_d(C)^{\natural}$   |
|----------------------------|----------|--|
| $k^{\bar{d}i}f^{2\bar{d}}$ | $i = -1$ | $-(q - q^{-1})^{2\bar{d}}q^{\bar{d}}a^{\bar{d}}b^{-2\bar{d}}$  |
|                            | $i = 0$  | $2(q - q^{-1})^{2\bar{d}}b^{-\bar{d}}c^{\bar{d}}$  |
|                            | $i = 1$  | $-(q - q^{-1})^{2\bar{d}}q^{\bar{d}}a^{-\bar{d}}c^{2\bar{d}}$  |
| $k^{\bar{d}i}f^{\bar{d}}$  | $i = -2$ | $-3(q - q^{-1})^{\bar{d}}q^{\bar{d}}a^{\bar{d}}b^{-2\bar{d}}$  |
|                            | $i = -1$ | $(q - q^{-1})^{\bar{d}}b^{-\bar{d}}(a^{\bar{d}}b^{-\bar{d}}T_d(\Lambda) + 4c^{\bar{d}} + c^{-\bar{d}})$                            |
|                            | $i = 0$  | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}a^{-\bar{d}}(a^{\bar{d}}b^{-\bar{d}}c^{\bar{d}}T_d(\Lambda) + a^{2\bar{d}} + 1 + c^{2\bar{d}})$ |
|                            | $i = 1$  | $(q - q^{-1})^{\bar{d}}b^{\bar{d}}c^{\bar{d}}$   |
| $k^{\bar{d}i}$             | $i = -3$ | $-3q^{\bar{d}}a^{\bar{d}}b^{-2\bar{d}}$  |
|                            | $i = -2$ | $2b^{-\bar{d}}(a^{\bar{d}}b^{-\bar{d}}T_d(\Lambda) + c^{\bar{d}} + c^{-\bar{d}})$  |
|                            | $i = -1$ | $-q^{\bar{d}}a^{\bar{d}}(a^{-\bar{d}}b^{-\bar{d}}(c^{\bar{d}} + c^{-\bar{d}})T_d(\Lambda) + a^{-2\bar{d}} + 2 + b^{-2\bar{d}})$    |
|                            | $i = 0$  | $(a^{\bar{d}} + a^{-\bar{d}})T_d(\Lambda) + (b^{\bar{d}} + b^{-\bar{d}})(c^{\bar{d}} + c^{-\bar{d}})$                              |
|                            | $i = 1$  | $-q^{\bar{d}}a^{-\bar{d}}$   |
| $k^{\bar{d}i}e^{\bar{d}}$  | $i = -3$ | $(q - q^{-1})^{\bar{d}}a^{\bar{d}}b^{-2\bar{d}}$   |
|                            | $i = -2$ | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}b^{-\bar{d}}c^{-\bar{d}}$   |
|                            | $i = -1$ | $(q - q^{-1})^{\bar{d}}a^{\bar{d}}$  |
|                            | $i = 0$  | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}b^{\bar{d}}c^{-\bar{d}}$  |

| coefficients               |          | $T_d(C)^{\natural}T_d(A)^{\natural}$   |
|----------------------------|----------|--|
| $k^{\bar{d}i}f^{\bar{d}}$  | $i = -2$ | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}a^{2\bar{d}}b^{-\bar{d}}$   |
|                            | $i = -1$ | $(q - q^{-1})^{\bar{d}}a^{\bar{d}}c^{\bar{d}}$   |
|                            | $i = 0$  | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}b^{-\bar{d}}$   |
|                            | $i = 1$  | $(q - q^{-1})^{\bar{d}}a^{-\bar{d}}c^{\bar{d}}$  |
| $k^{\bar{d}i}$             | $i = -3$ | $-3q^{\bar{d}}a^{2\bar{d}}b^{-\bar{d}}$  |
|                            | $i = -2$ | $2a^{\bar{d}}(a^{\bar{d}}b^{-\bar{d}}T_d(\Lambda) + c^{\bar{d}} + c^{-\bar{d}})$   |
|                            | $i = -1$ | $-q^{\bar{d}}b^{-\bar{d}}(a^{\bar{d}}b^{\bar{d}}(c + c^{-\bar{d}})T_d(\Lambda) + a^{2\bar{d}} + 2 + b^{2\bar{d}})$       |
|                            | $i = 0$  | $(b + b^{-\bar{d}})T_d(\Lambda) + (a^{\bar{d}} + a^{-\bar{d}})(c^{\bar{d}} + c^{-\bar{d}})$                              |
|                            | $i = 1$  | $-q^{\bar{d}}b^{\bar{d}}$  |
| $k^{\bar{d}i}e^{\bar{d}}$  | $i = -3$ | $3(q - q^{-1})^{\bar{d}}a^{2\bar{d}}b^{-\bar{d}}$  |
|                            | $i = -2$ | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}a^{\bar{d}}(a^{\bar{d}}b^{-\bar{d}}T_d(\Lambda) + c^{\bar{d}} + 4c^{-\bar{d}})$       |
|                            | $i = -1$ | $(q - q^{-1})^{\bar{d}}b^{\bar{d}}(a^{\bar{d}}b^{-\bar{d}}c^{-\bar{d}}T_d(\Lambda) + b^{-2\bar{d}} + 1 + c^{-2\bar{d}})$ |
|                            | $i = 0$  | $-(q - q^{-1})^{\bar{d}}q^{\bar{d}}a^{-\bar{d}}c^{-\bar{d}}$   |
| $k^{\bar{d}i}e^{2\bar{d}}$ | $i = -3$ | $-(q - q^{-1})^{2\bar{d}}q^{\bar{d}}a^{2\bar{d}}b^{-\bar{d}}$  |
|                            | $i = -2$ | $2(q - q^{-1})^{2\bar{d}}a^{\bar{d}}c^{-\bar{d}}$  |
|                            | $i = -1$ | $-(q - q^{-1})^{2\bar{d}}q^{\bar{d}}b^{\bar{d}}c^{-2\bar{d}}$  |

| coefficients               |          | $T_d(A)^{\natural}T_d(B)^{\natural}T_d(C)^{\natural}$  |
|----------------------------|----------|--|
| $k^{\vec{d}i}f^{2\vec{d}}$ | $i = -2$ | $-(q - q^{-1})^{2\vec{d}}q^{\vec{d}}a^{2\vec{d}}b^{-2\vec{d}}$   |
|                            | $i = -1$ | $2(q - q^{-1})^{2\vec{d}}a^{\vec{d}}b^{-\vec{d}}c^{\vec{d}}$   |
|                            | $i = 0$  | $-(q - q^{-1})^{2\vec{d}}q^{\vec{d}}(b^{-2\vec{d}} + c^{2\vec{d}})$  |
|                            | $i = 1$  | $2(q - q^{-1})^{2\vec{d}}a^{-\vec{d}}b^{-\vec{d}}c^{\vec{d}}$  |
|                            | $i = 2$  | $-(q - q^{-1})^{2\vec{d}}q^{\vec{d}}a^{-2\vec{d}}c^{2\vec{d}}$   |
| $k^{\vec{d}i}f^{\vec{d}}$  | $i = -3$ | $-4(q - q^{-1})^{\vec{d}}q^{\vec{d}}a^{2\vec{d}}b^{-2\vec{d}}$   |
|                            | $i = -2$ | $2(q - q^{-1})^{\vec{d}}a^{\vec{d}}b^{-\vec{d}}(a^{\vec{d}}b^{-\vec{d}}T_d(\Lambda) + 3c^{\vec{d}} + c^{-\vec{d}})$  |
|                            | $i = -1$ | $-(q - q^{-1})^{\vec{d}}q^{\vec{d}}(a^{\vec{d}}b^{-\vec{d}}(3c^{\vec{d}} + c^{-\vec{d}})T_d(\Lambda) + (a^{2\vec{d}} + 3)(b^{-2\vec{d}} + 1) + 2c^{2\vec{d}})$   |
|                            | $i = 0$  | $(q - q^{-1})^{\vec{d}}((b^{-2\vec{d}} + 2 + c^{2\vec{d}})T_d(\Lambda) + b^{-\vec{d}}c^{\vec{d}}(a^{\vec{d}} + a^{-\vec{d}})(b^{2\vec{d}} + 2 + c^{-2\vec{d}}) + 2a^{-\vec{d}}b^{-\vec{d}}c^{\vec{d}})$                            |
|                            | $i = 1$  | $-(q - q^{-1})^{\vec{d}}q^{\vec{d}}(a^{-\vec{d}}c^{\vec{d}}(b^{\vec{d}} + b^{-\vec{d}})T_d(\Lambda) + (a^{-2\vec{d}} + 1)(c^{2\vec{d}} + 1) + 2)$  |
|                            | $i = 2$  | $2(q - q^{-1})^{\vec{d}}a^{-\vec{d}}b^{\vec{d}}c^{\vec{d}}$  |
| $k^{\vec{d}i}$             | $i = -4$ | $-6q^{\vec{d}}a^{2\vec{d}}b^{-2\vec{d}}$   |
|                            | $i = -3$ | $6a^{\vec{d}}b^{-\vec{d}}(a^{\vec{d}}b^{-\vec{d}}T_d(\Lambda) + c^{\vec{d}} + c^{-\vec{d}})$   |
|                            | $i = -2$ | $-q^{\vec{d}}(a^{2\vec{d}}b^{-2\vec{d}}T_d(\Lambda)^2 + 6a^{\vec{d}}b^{-\vec{d}}(c^{\vec{d}} + c^{-\vec{d}})T_d(\Lambda) + (a^{2\vec{d}} + 3)(b^{-2\vec{d}} + 3) + 3a^{2\vec{d}}b^{-2\vec{d}} + c^{2\vec{d}} - 3 + c^{-2\vec{d}})$ |
|                            | $i = -1$ | $a^{\vec{d}}b^{-\vec{d}}(c^{\vec{d}} + c^{-\vec{d}})(T_d(\Lambda)^2 + (a^{-2\vec{d}} + 2)(b^{2\vec{d}} + 2) + 1) + ((a^{2\vec{d}} + 2)(b^{-2\vec{d}} + 2) + c^{2\vec{d}} + 3 + c^{-2\vec{d}})T_d(\Lambda)$                         |
|                            | $i = 0$  | $-q^{\vec{d}}(T_d(\Lambda)^2 + (a^{\vec{d}} + a^{-\vec{d}})(b^{\vec{d}} + b^{-\vec{d}})(c^{\vec{d}} + c^{-\vec{d}})T_d(\Lambda) + a^{2\vec{d}} + b^{2\vec{d}} + c^{2\vec{d}} + 8 + a^{-2\vec{d}} + b^{-2\vec{d}} + c^{-2\vec{d}})$ |
|                            | $i = 1$  | $(a^{-2\vec{d}} + 2 + b^{2\vec{d}})T_d(\Lambda) + a^{-\vec{d}}b^{\vec{d}}(a^{2\vec{d}} + 2 + b^{-2\vec{d}})(c^{\vec{d}} + c^{-\vec{d}})$   |
|                            | $i = 2$  | $-q^{\vec{d}}(a^{-2\vec{d}} + b^{2\vec{d}})$   |
| $k^{\vec{d}i}e^{\vec{d}}$  | $i = -4$ | $4(q - q^{-1})^{\vec{d}}a^{2\vec{d}}b^{-2\vec{d}}$   |
|                            | $i = -3$ | $-2(q - q^{-1})^{\vec{d}}q^{\vec{d}}a^{\vec{d}}b^{-\vec{d}}(a^{\vec{d}}b^{-\vec{d}}T_d(\Lambda) + c^{\vec{d}} + 3c^{-\vec{d}})$  |
|                            | $i = -2$ | $(q - q^{-1})^{\vec{d}}(a^{\vec{d}}b^{-\vec{d}}(c^{\vec{d}} + 3c^{-\vec{d}})T_d(\Lambda) + (a^{2\vec{d}} + 1)(b^{-2\vec{d}} + 3) + 2c^{-2\vec{d}})$  |
|                            | $i = -1$ | $-(q - q^{-1})^{\vec{d}}q^{\vec{d}}((a^{2\vec{d}} + 2 + c^{-2\vec{d}})T_d(\Lambda) + a^{\vec{d}}c^{-\vec{d}}(b^{\vec{d}} + b^{-\vec{d}})(a^{-2\vec{d}} + 2 + c^{2\vec{d}}) + 2a^{\vec{d}}b^{\vec{d}}c^{-\vec{d}})$                 |
|                            | $i = 0$  | $(q - q^{-1})^{\vec{d}}(b^{\vec{d}}c^{-\vec{d}}(a^{\vec{d}} + a^{-\vec{d}})T_d(\Lambda) + (b^{2\vec{d}} + 1)(c^{-2\vec{d}} + 1) + 2)$  |
|                            | $i = 1$  | $-2(q - q^{-1})^{\vec{d}}q^{\vec{d}}a^{-\vec{d}}b^{\vec{d}}c^{-\vec{d}}$   |
| $k^{\vec{d}i}e^{2\vec{d}}$ | $i = -4$ | $-(q - q^{-1})^{2\vec{d}}q^{\vec{d}}a^{2\vec{d}}b^{-2\vec{d}}$   |
|                            | $i = -3$ | $2(q - q^{-1})^{2\vec{d}}a^{\vec{d}}b^{-\vec{d}}c^{-\vec{d}}$  |
|                            | $i = -2$ | $-(q - q^{-1})^{2\vec{d}}q^{\vec{d}}(a^{2\vec{d}} + c^{-2\vec{d}})$  |
|                            | $i = -1$ | $2(q - q^{-1})^{2\vec{d}}a^{\vec{d}}b^{\vec{d}}c^{-\vec{d}}$   |
|                            | $i = 0$  | $-(q - q^{-1})^{2\vec{d}}q^{\vec{d}}b^{2\vec{d}}c^{-2\vec{d}}$   |

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